

**CATATAN KULIAH
(*CLASSNOTES*)**

**DASAR SISTEM KENDALI
(*Basic Control Systems*)**

Rhiza S. Sadjad

2015

Dasar Sistem Kendali

* Pengenalan "SISTEM KENDALI"

1995 Konsorsium Ilmu² Teknik

* Teknik Elektro (Electrical Engineering):

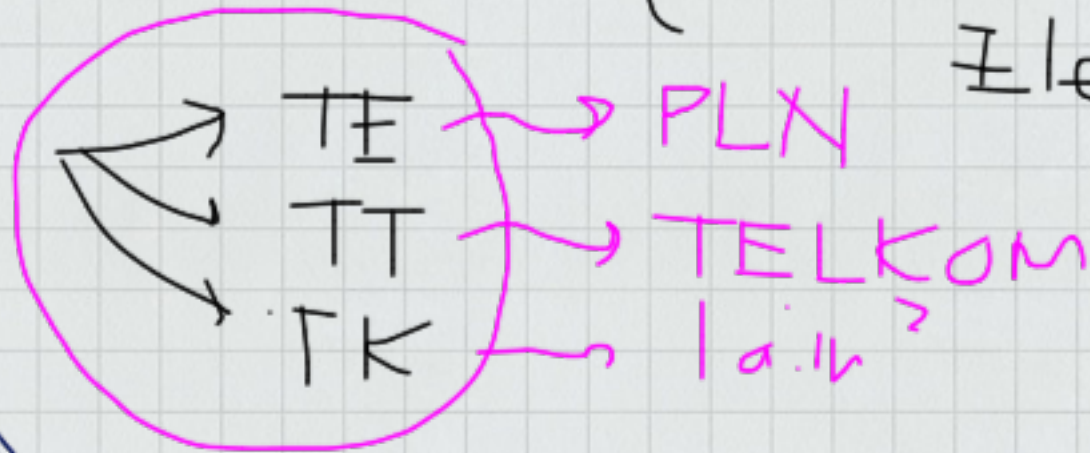
IEEE	* Teknik Energi (listrik)	} 1960 → IEEE
IRE	* Teknik Telekomunikasi	
	* Teknik Elektronika	} TK KE
→ ⊗	* Teknik Kendali / Kontrol	
	* Teknik <u>Komputer</u> ≈	

1996 * Teknik Informatika
→ Masyarakat Sistem Kendali Indonesia

MASDALI

UNHAS : 1963 → Teknik Elektro di UNHAS
1980 → TTL (Teknik Tenaga Listrik)
 → TTE (Teknik Telekomunikasi Elektronik)

1995



2015

"Control" (1996) → "KENDALI"

Control Engineering

Control System

→ Teknik Kendali

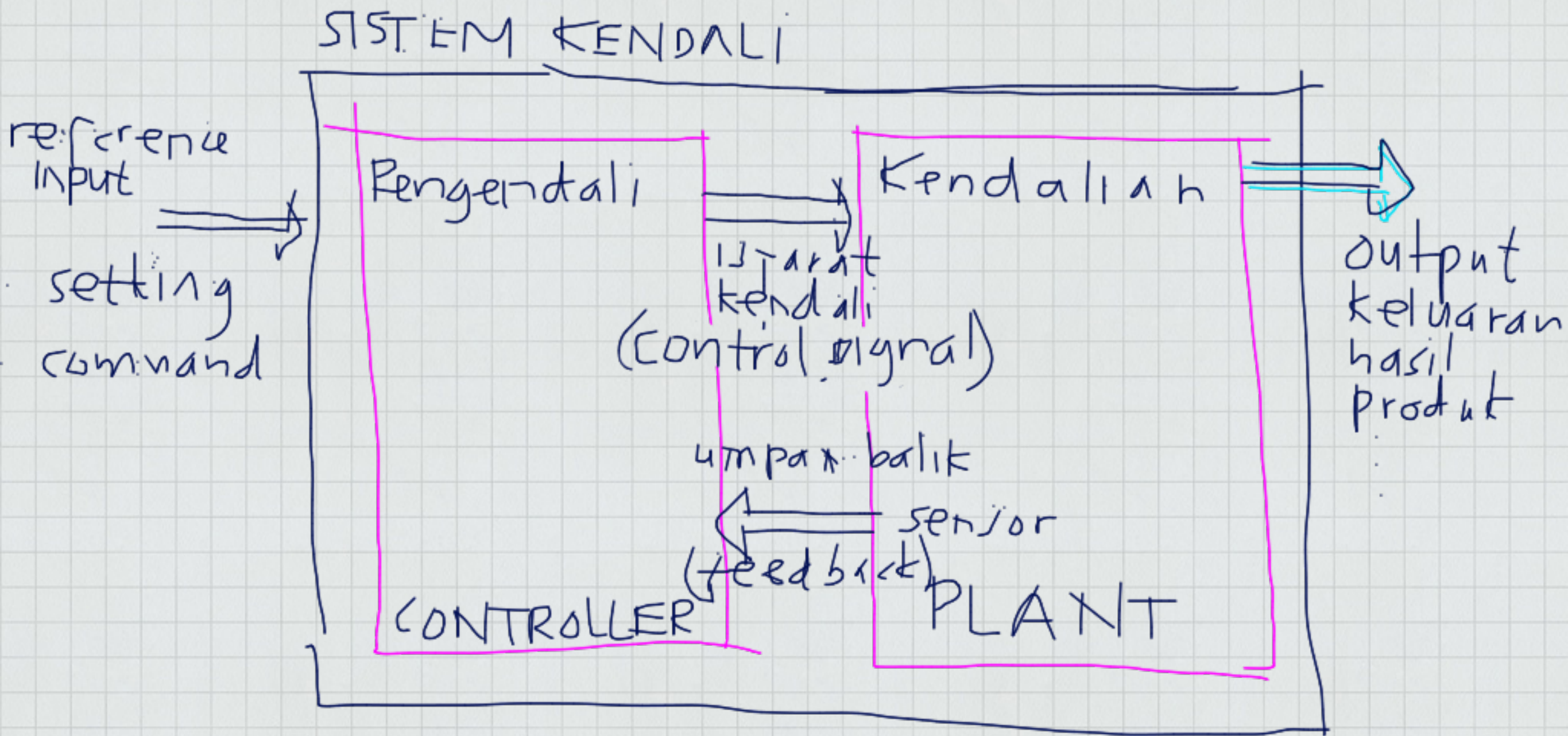
- * Teknik Pengaturan
- * Teknik Mengatur
- * Teknik Kontrol
- * Teknik Pengendalian

* SISTEM KENDALI (Control System)

[sistem kontrol
sistem pengaturan
sistem pengendalian] $\Leftarrow$

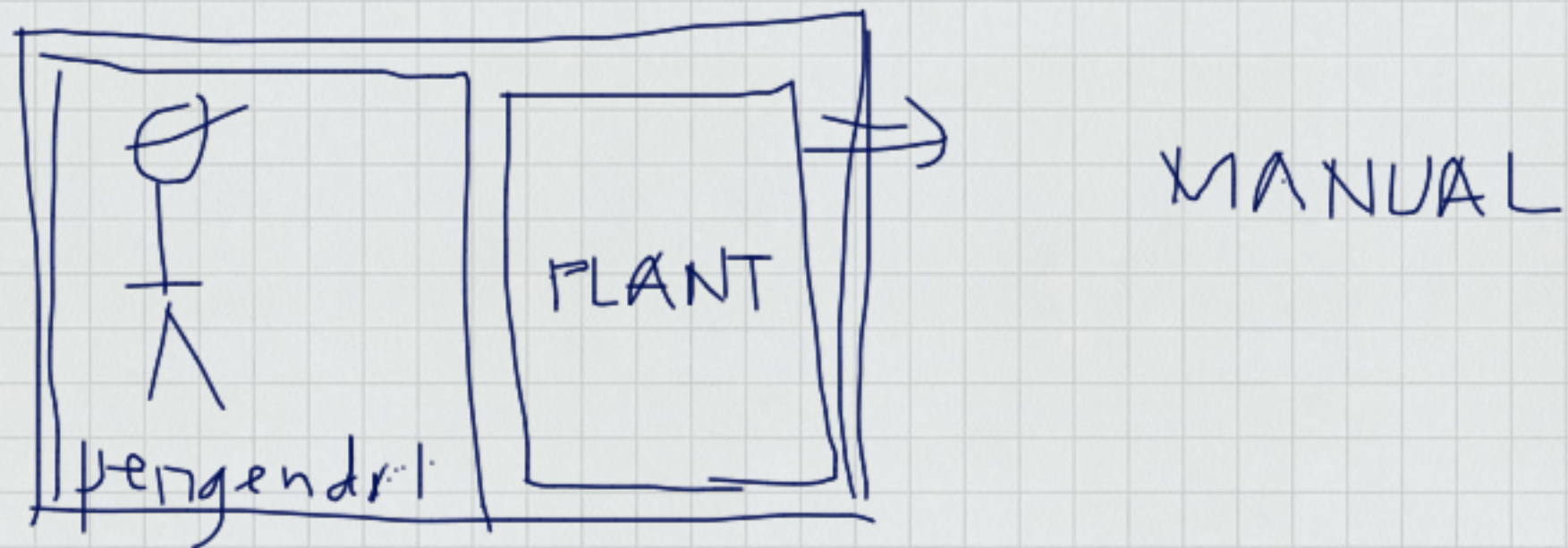
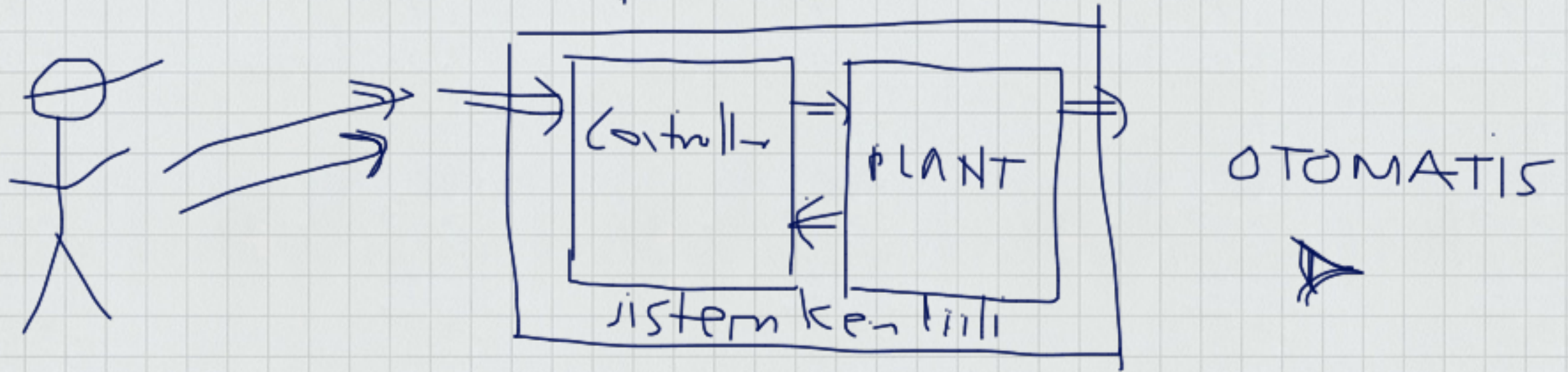
Sistem Kendali adalah: sistem apa saja yang mempunyai 2 (dua) bagian:

- sistem apa saja
- (1) Yang dikendalikan (KENDALIAN) (PLANT)
 - (2) Yang mengendalikan (PENGENDALI) (CONTROLLER)



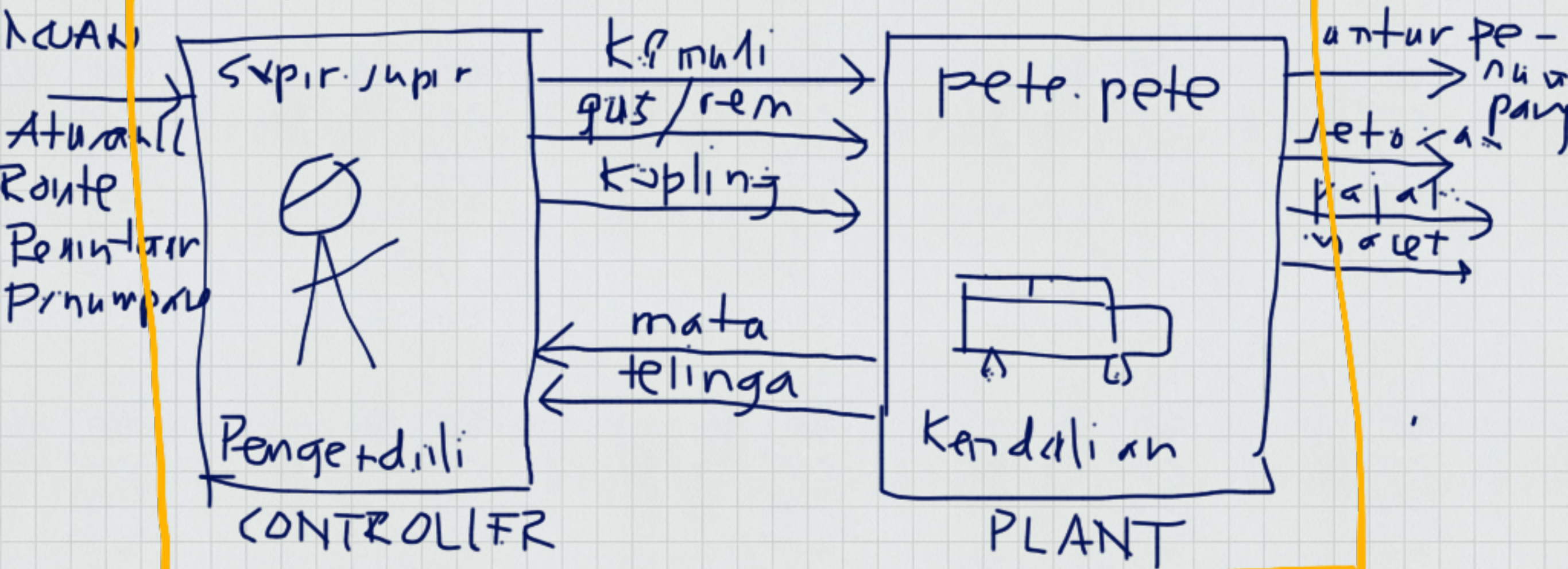
"Sistem Kendali Daur Terbuka" (open loop Control System)
"Sistem Kendali Daur Tertutup" (Closed loop Control System)

Sistem Kendali OTOMATIS x MANUAL (Automatic Control Systems)

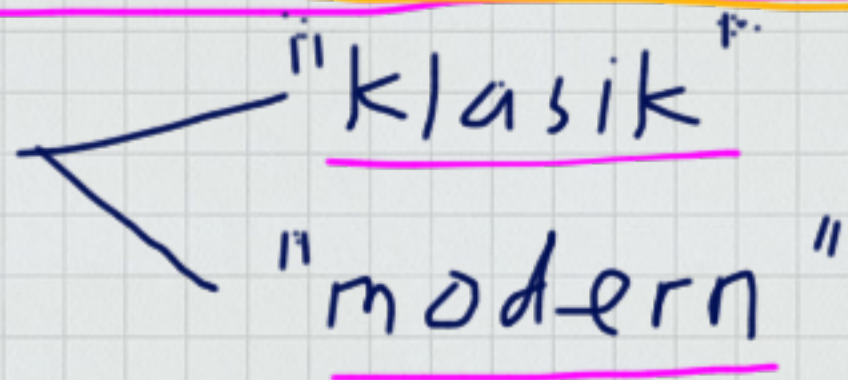


* SISTEM OPERASI "Pete. Pete"

SISTEM KENDALI MANUAL



BAB II. Alat-alat Matematik

Teori Kendali 
"klasik"
"modern"

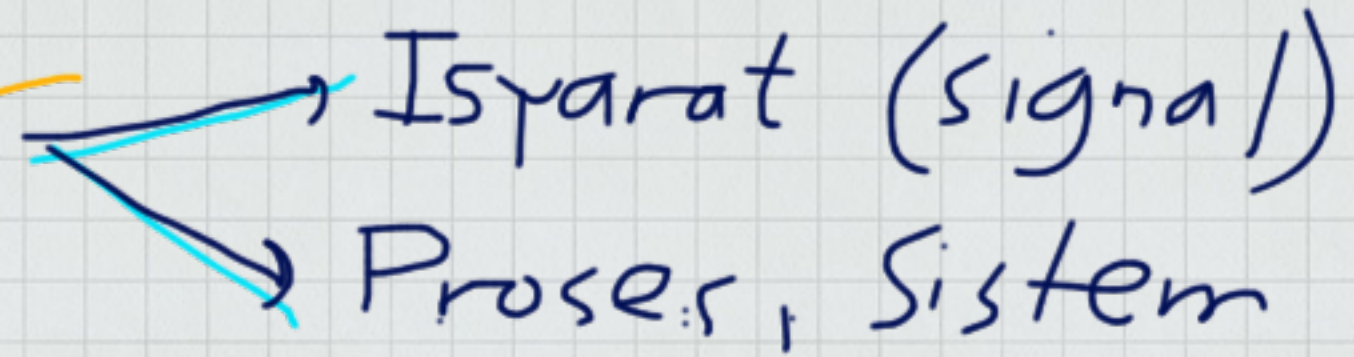
* Alat Matematik untuk Teori Kendali
KLASIK :

⇒ { * Bagan Kotak dan Aljabar Bagan Kotak
* Model Nisbah Alih dan Transformasi
LAPLACE

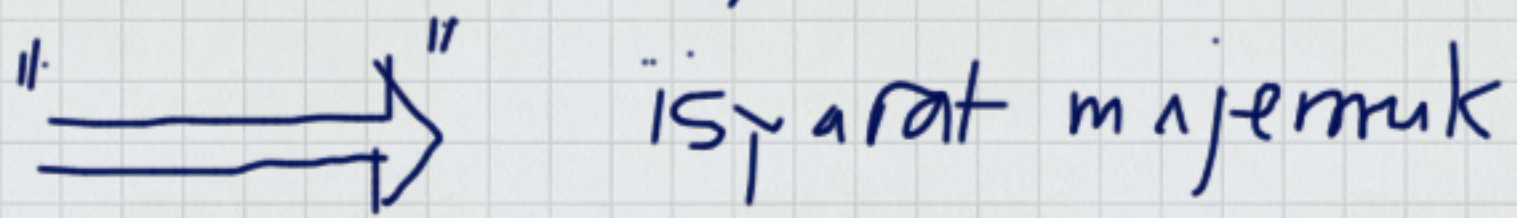
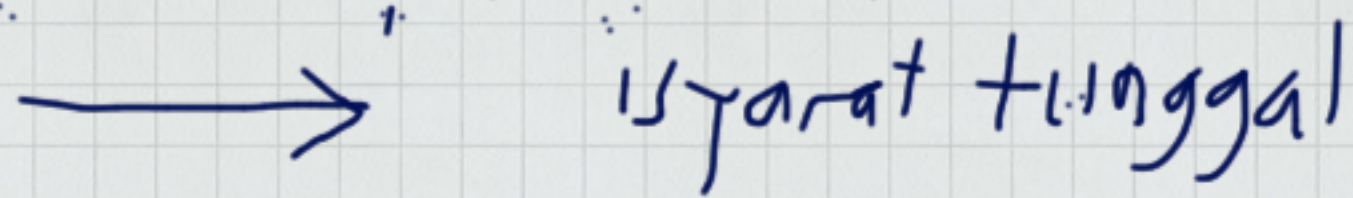
MODERN:

{ * Matrix dan Aljabar Linier

* BAGAN KOTAK (Block Diagram)



⇒ Notasi Isyarat (signal)



sumber
(source)

arah propagasi
(destination)

label isyarat

- * Nama ⇒ gelombang radio, tegangan,
- * Fungsi ⇒ $x(t)$, $\psi(k)$, $V_u(j\omega)$, $G(s)$

$x(t)$ = isyarat x yang berubah dengan
(sebagai fungsi dari) $t = \text{time}$
(waktu)

$Y(k)$ = isyarat Y yang berubah secara
diskontinu, sesuai urutan $k = 0, 1, 2, 3,$

"clock" "detak"

$Y(0)$: kondisi awal

$\rightarrow Y(1)$
 $\rightarrow Y(2)$

$V_i(j\omega)$ = isyarat V (= voltage, tegangan) masukan
($i = \text{input}$), yang berubah sebagai

fungsi dari $j\omega$, $j = \sqrt{-1}$ $f = \frac{1}{T}$

" ω " = omega

" Ω " = OMEGA

$\omega = 2\pi f$
frekuensi

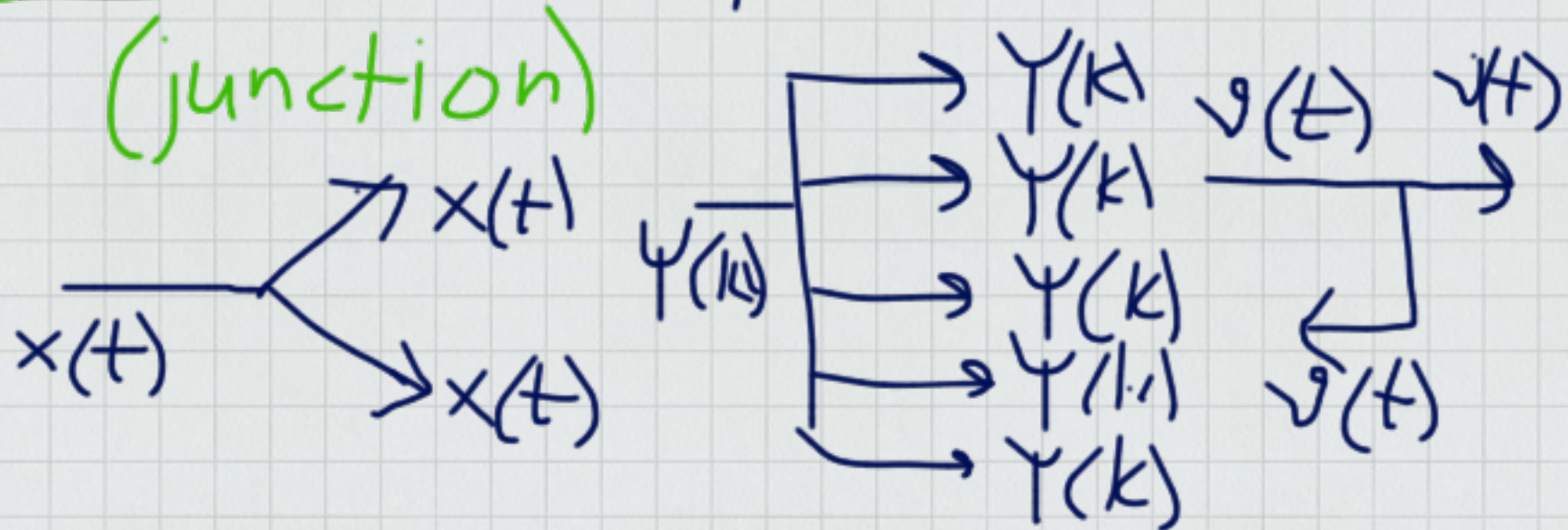
[Hz]

$\Pi : PI$

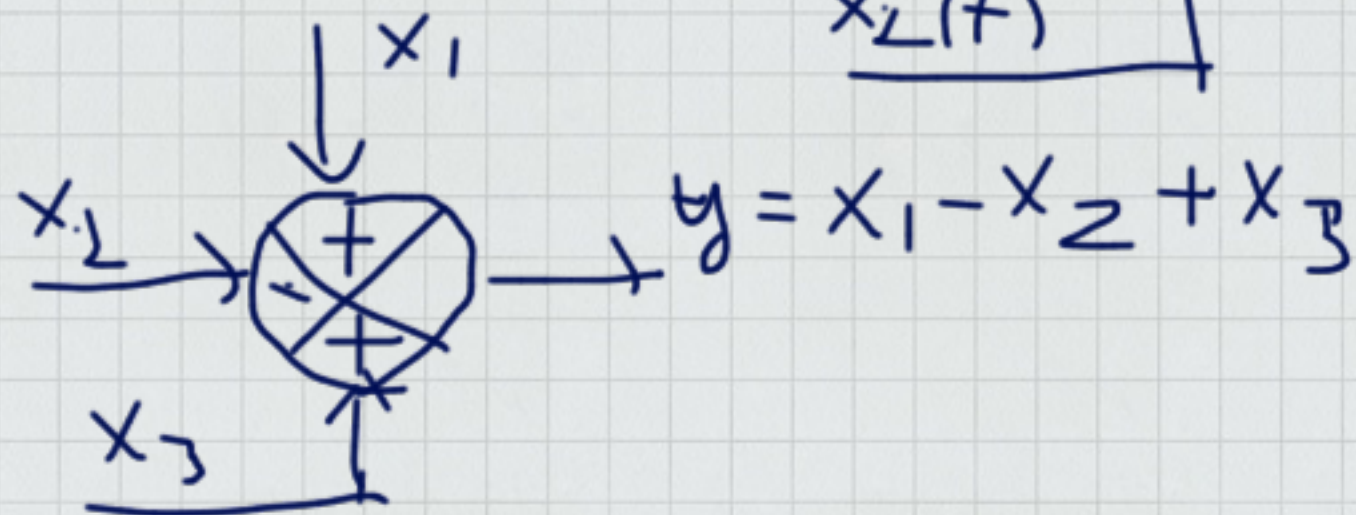
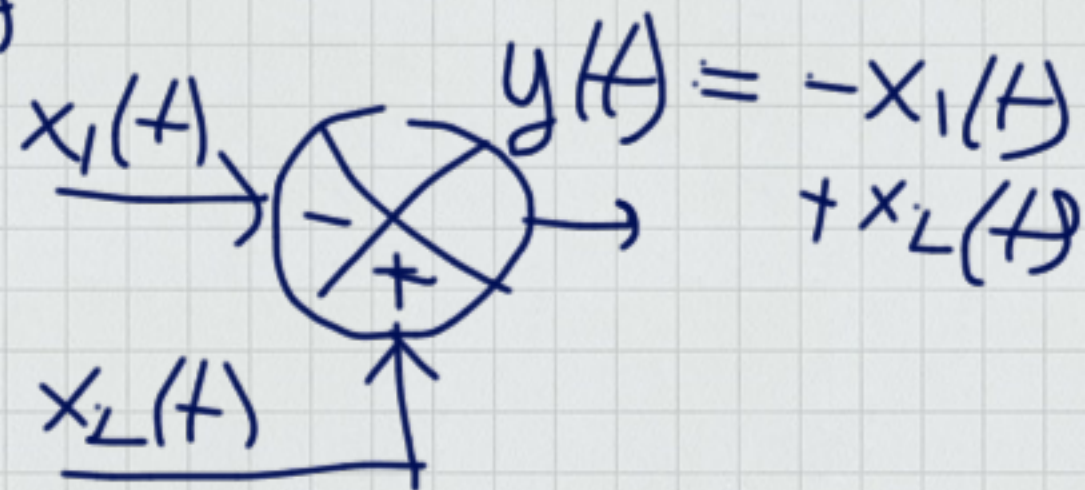
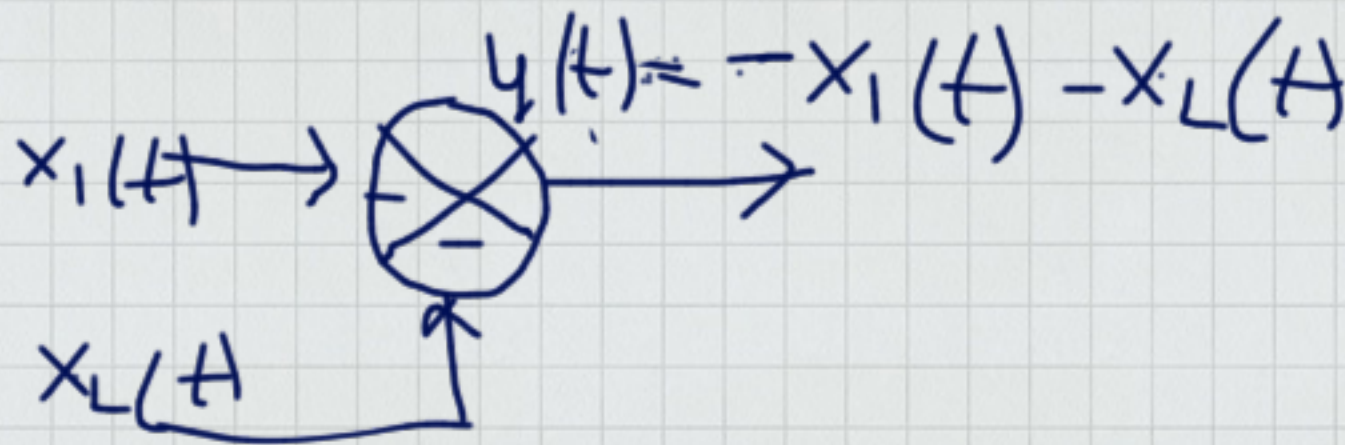
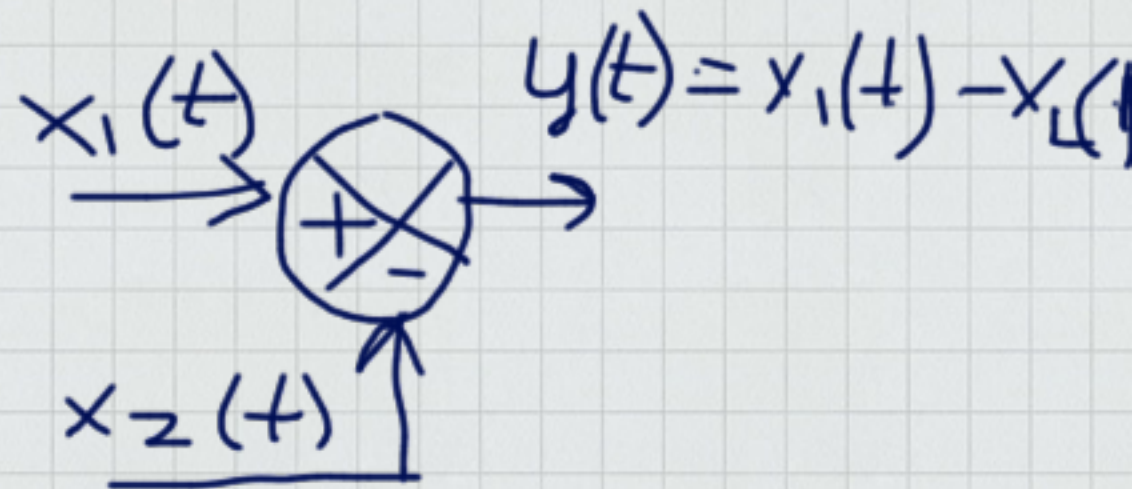
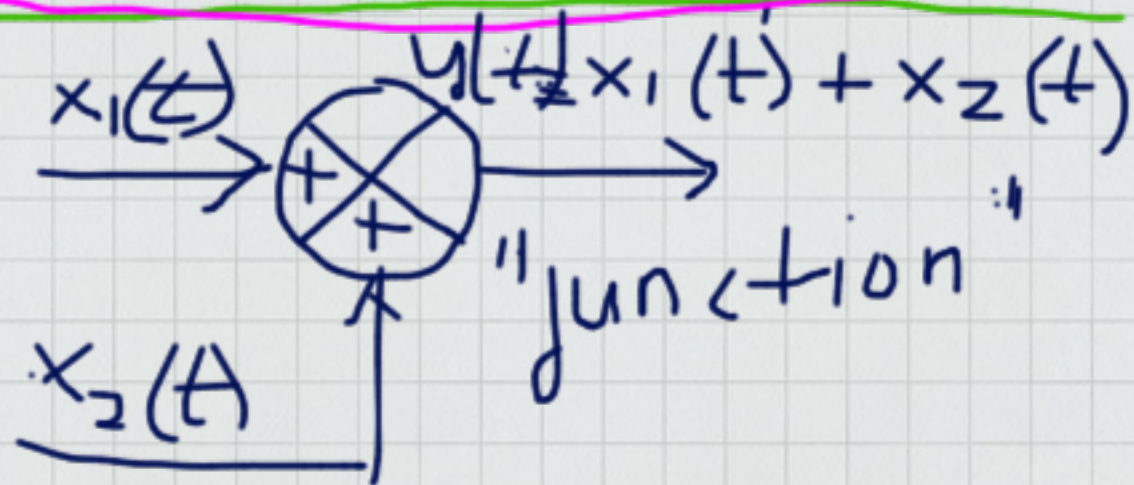
* $G(s)$ = Isyarat G yang berubah dengan s
 s = peubah Laplace (kompleks)

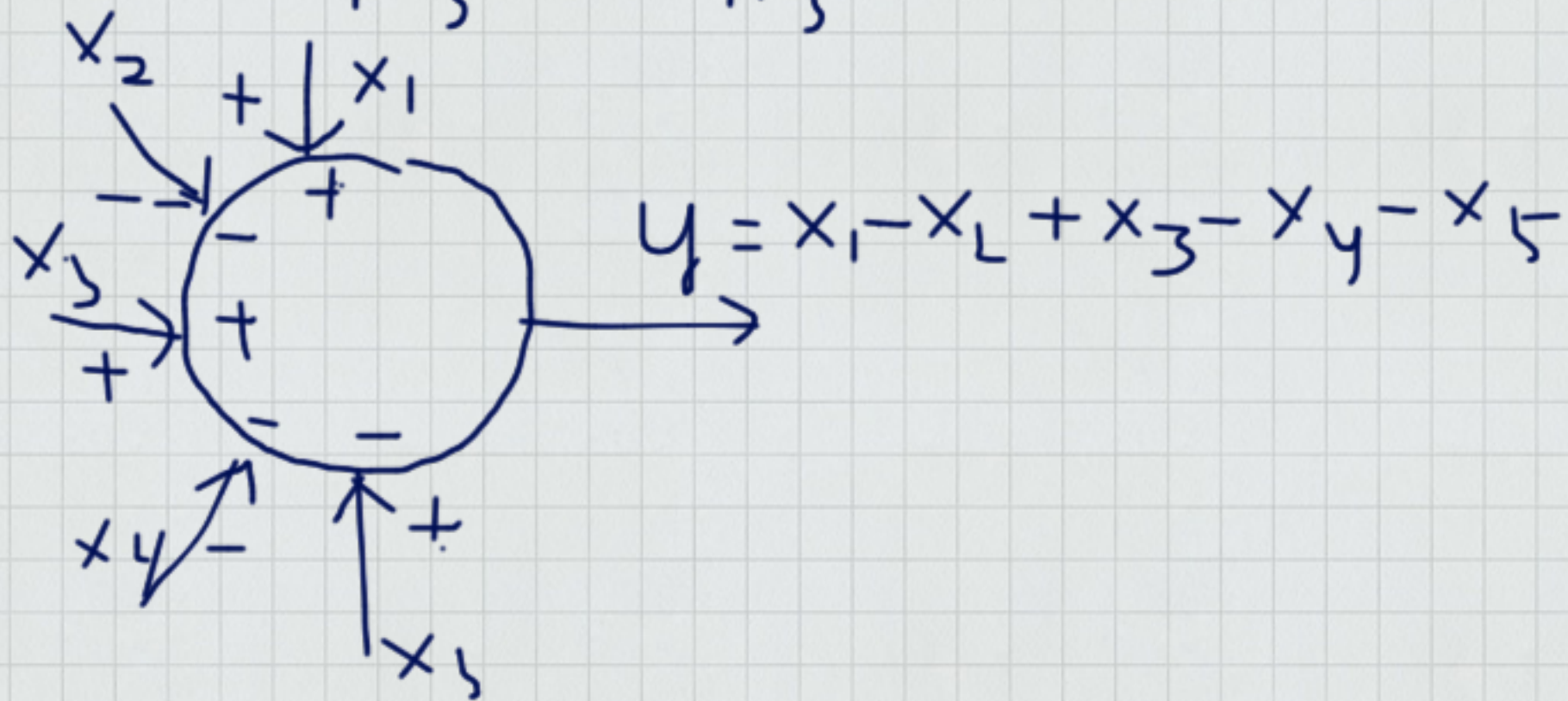
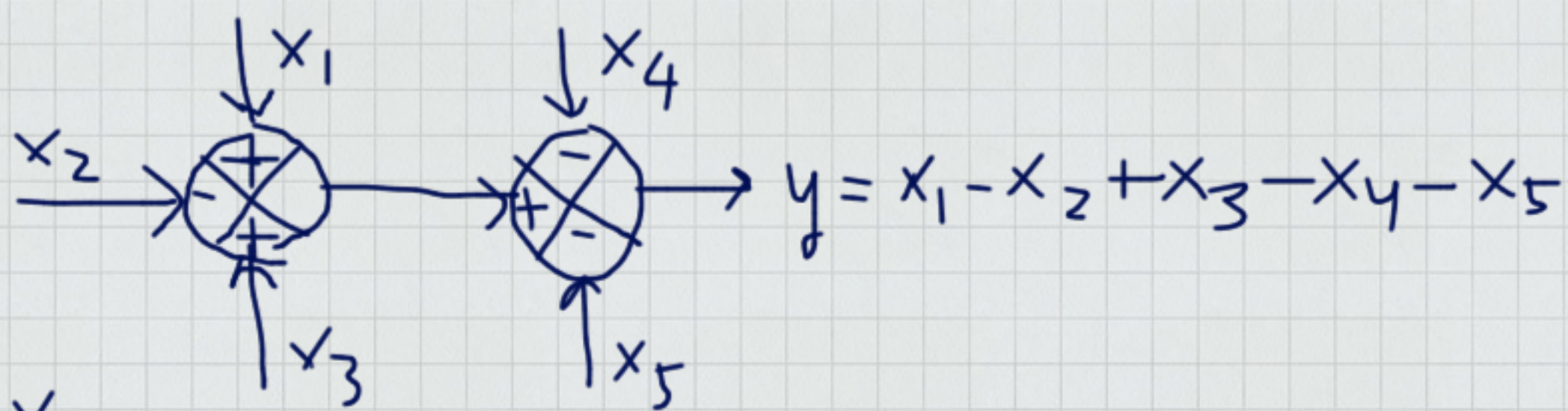
* Percabangan dan Pertemuan Isyarat
 (branch) (junction)

⇒ Percabangan:



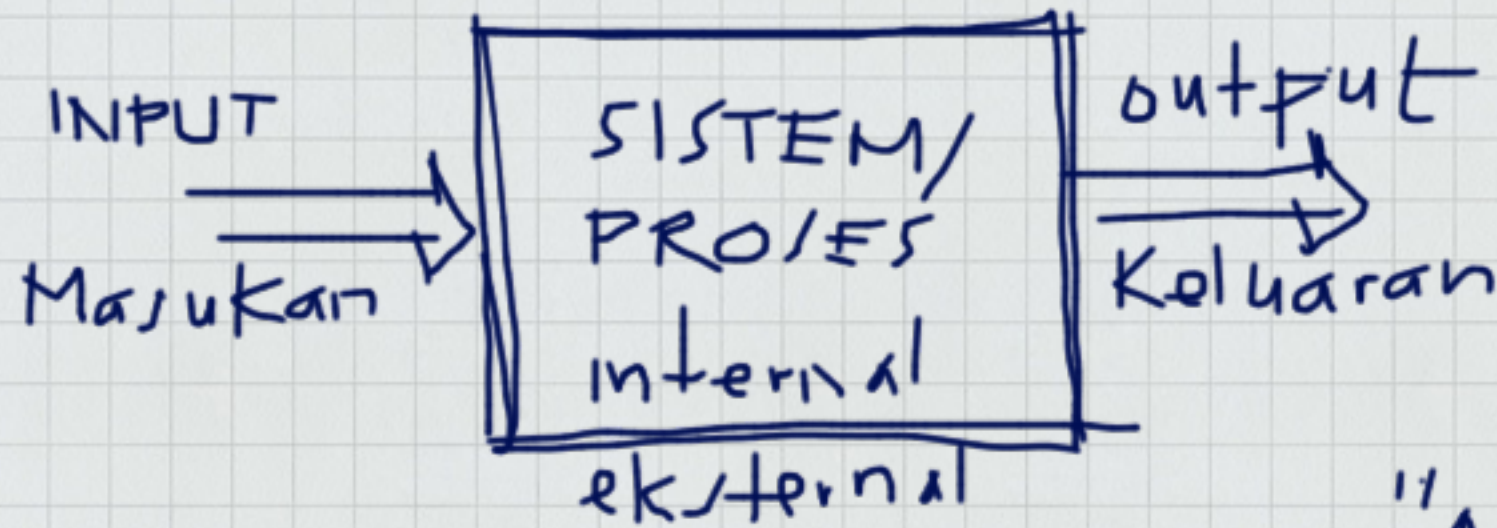
⇒ Pertemuan Isyarat





* Proses, Sistem

= 4-persegi panjang
- bujur sangkar

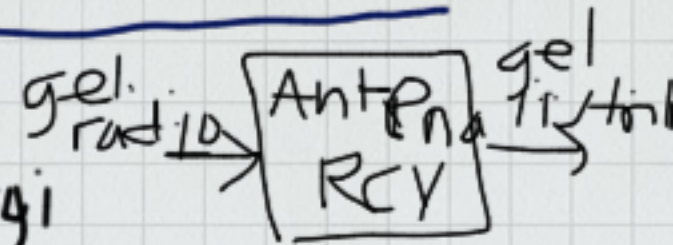
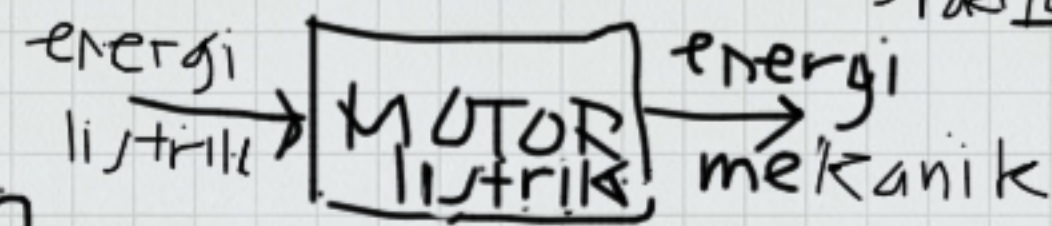
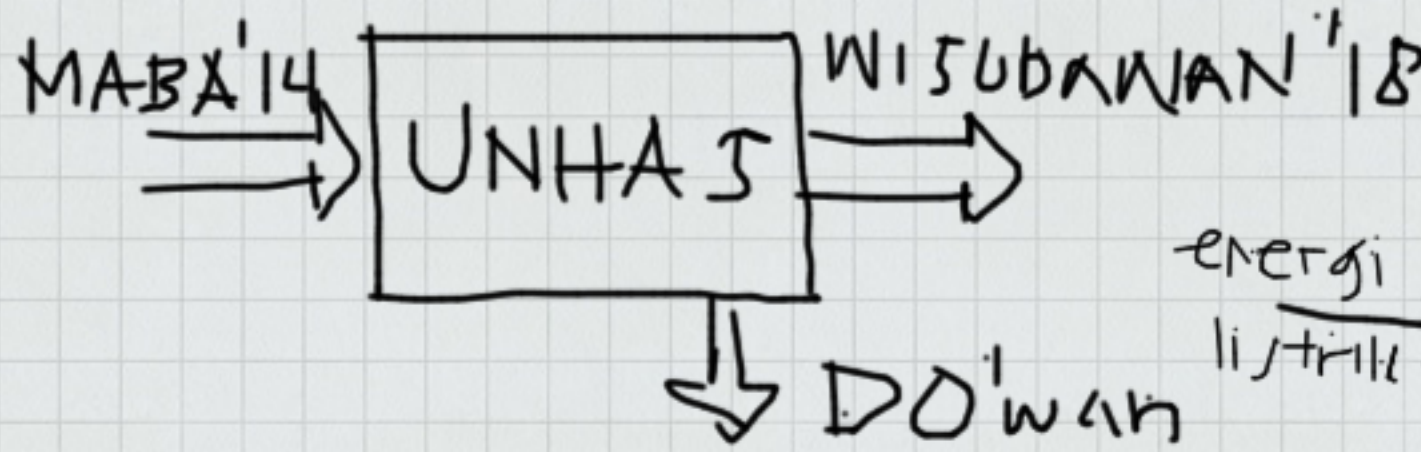


Notasi Proses/Sistem, contoh:

* Kata-kata/kalimat:

"gangguan" (disturbance)
"derau" (noise)
"lingkungan" (environment)

"fisik" ← "ambient"

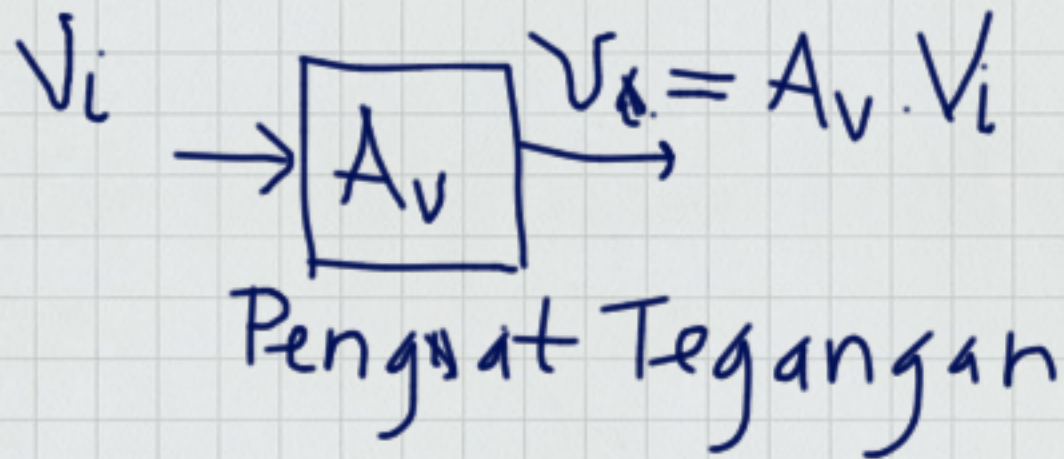
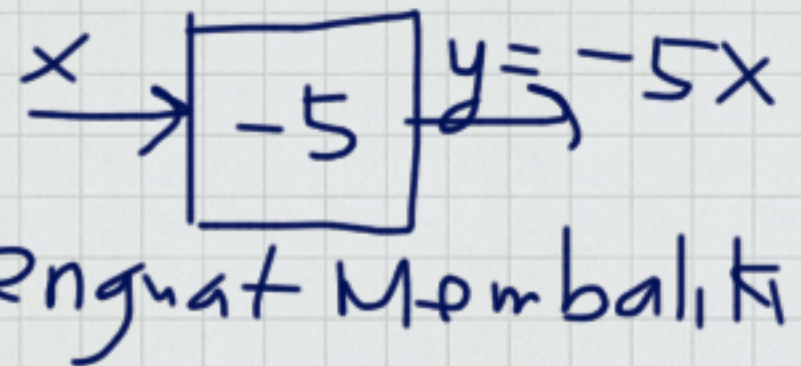
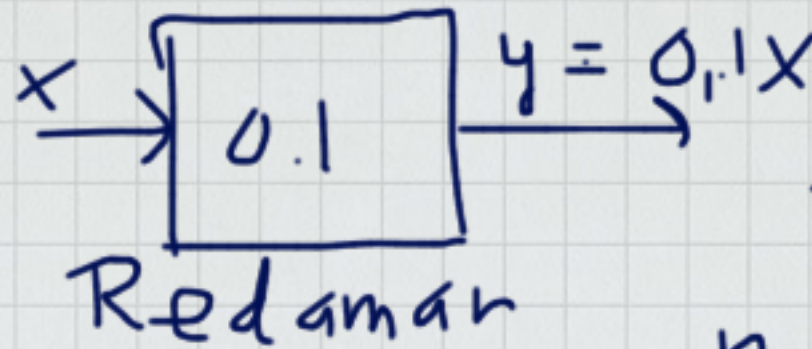
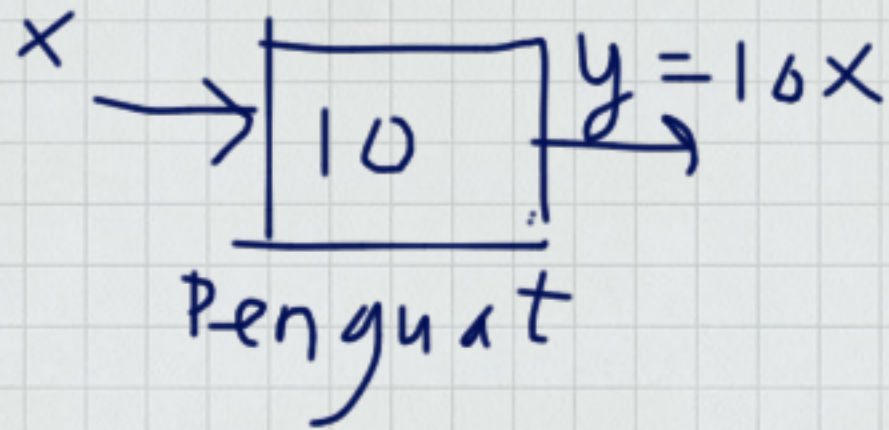


* Huruf/Angka:

$K > 1 \rightarrow$ Penguat/Amplifier

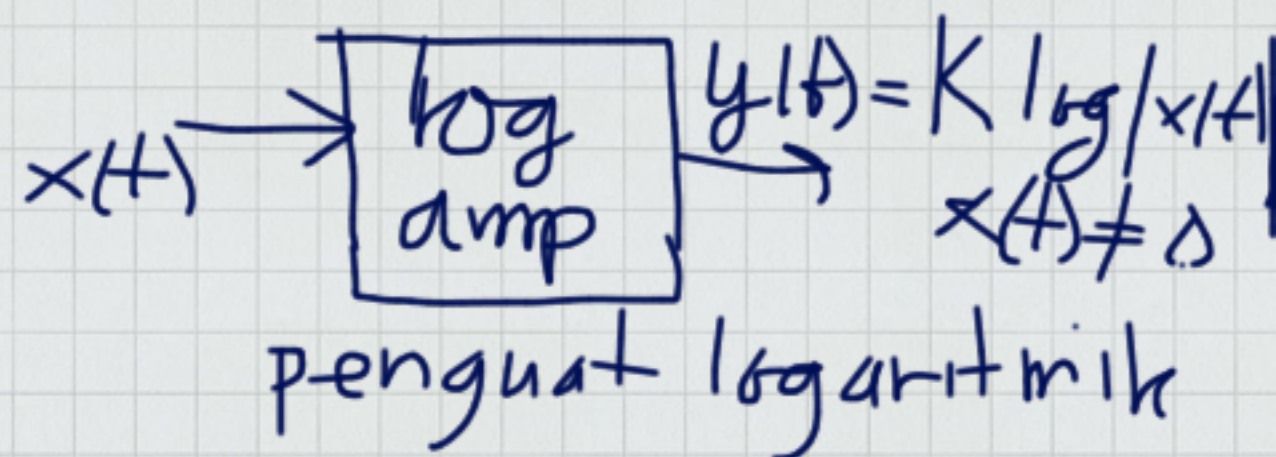
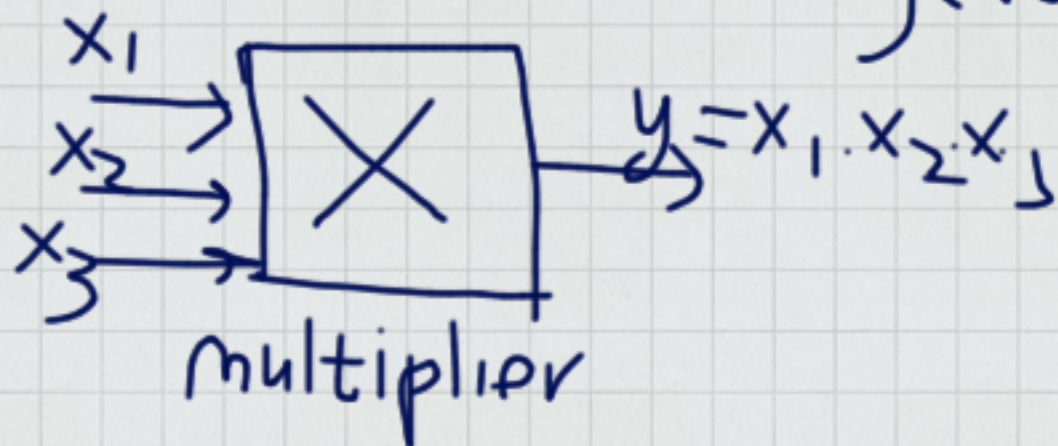
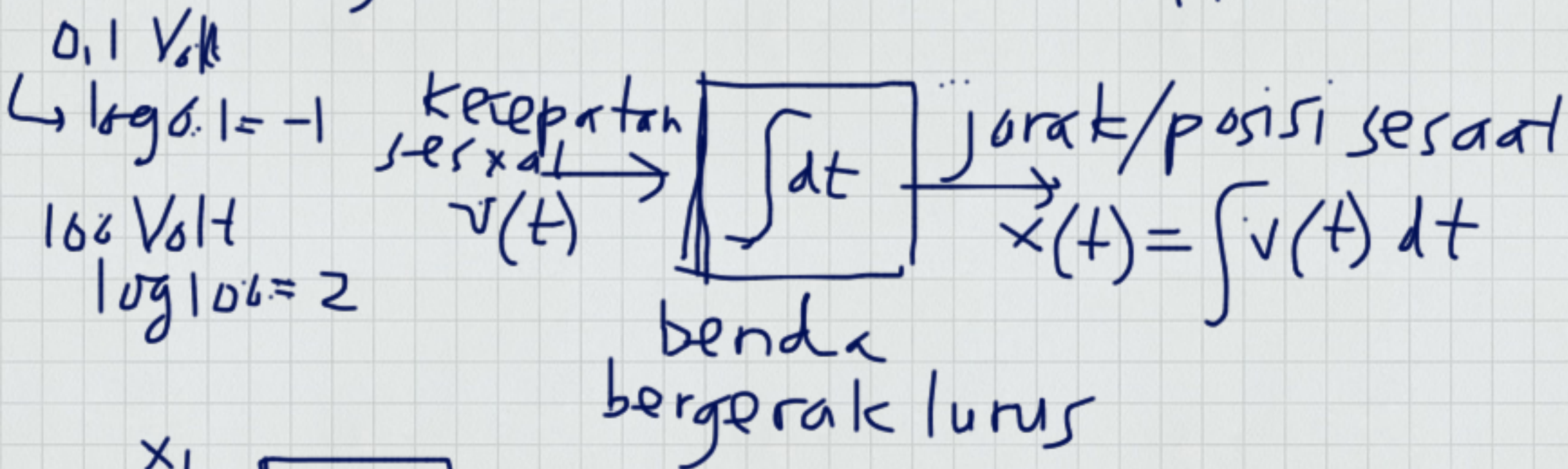
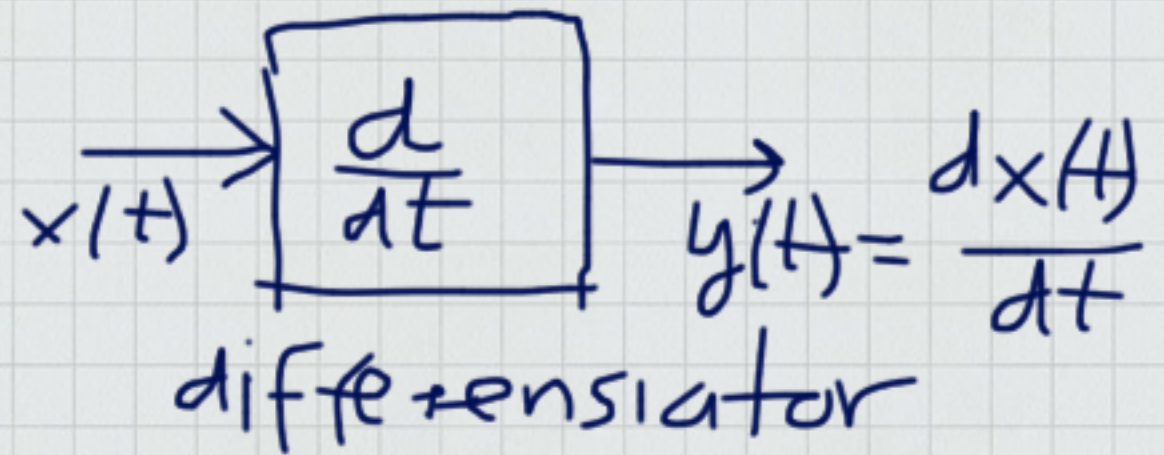
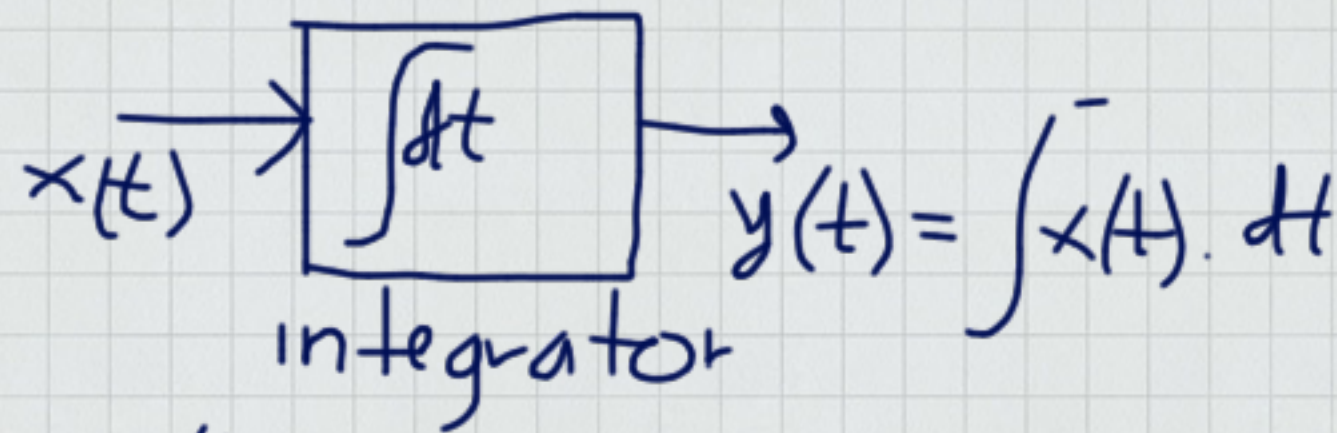
$0 < K < 1 \rightarrow$ Redaman/Attenuator

$K < 0 \rightarrow$ Penguat Membalik/
Inverter/Inverting
Amplifier

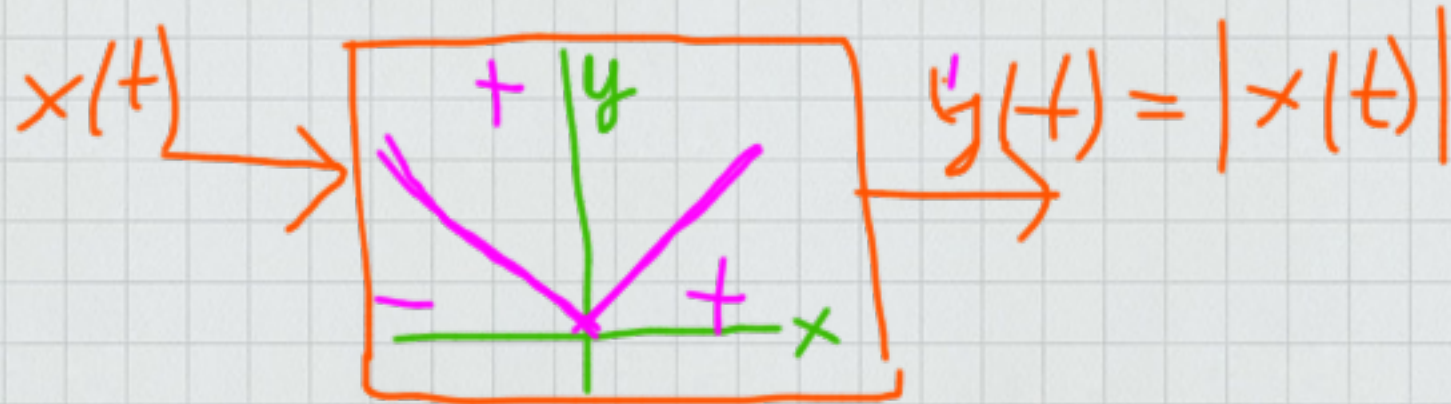


$A_v =$ "bati" = gain
penguatan $\leftarrow$
pelemahan $\rightarrow$
attenuation

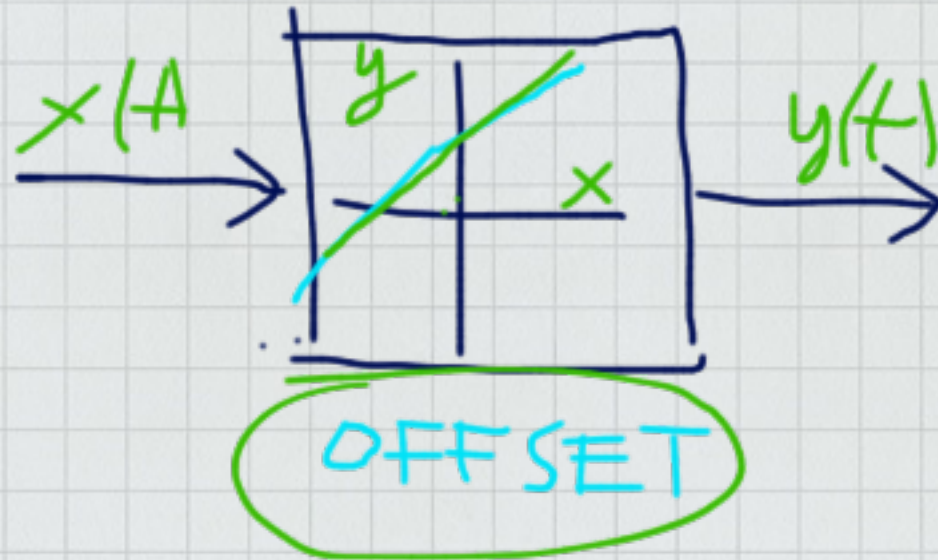
* Operasi Matematika



* Watak Alih (Transfer Characteristic)



Penyearah
(Rectifier)

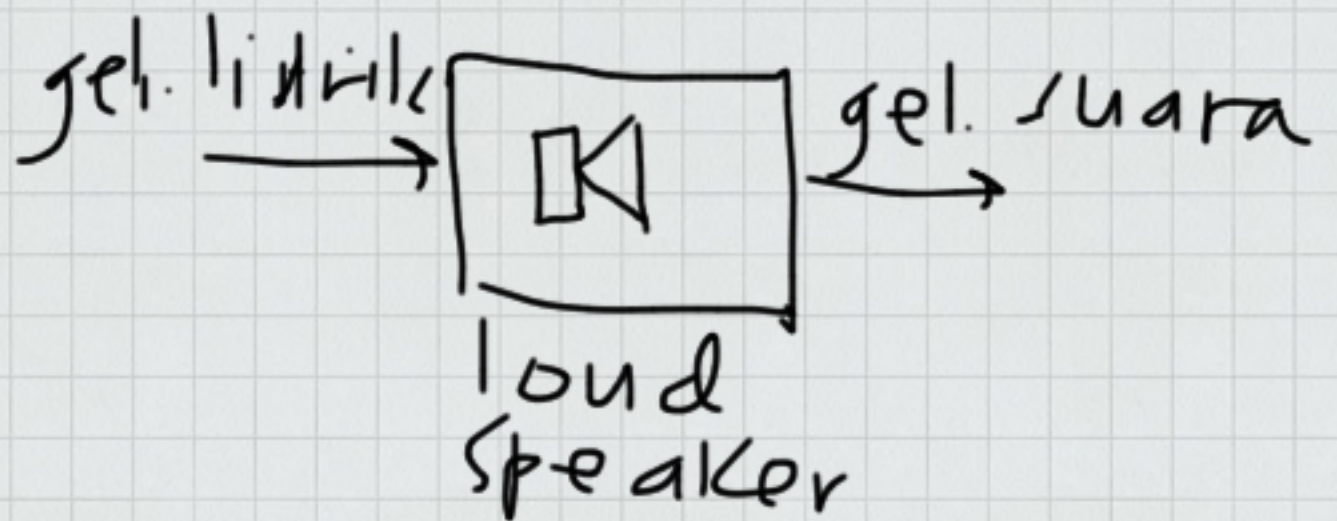
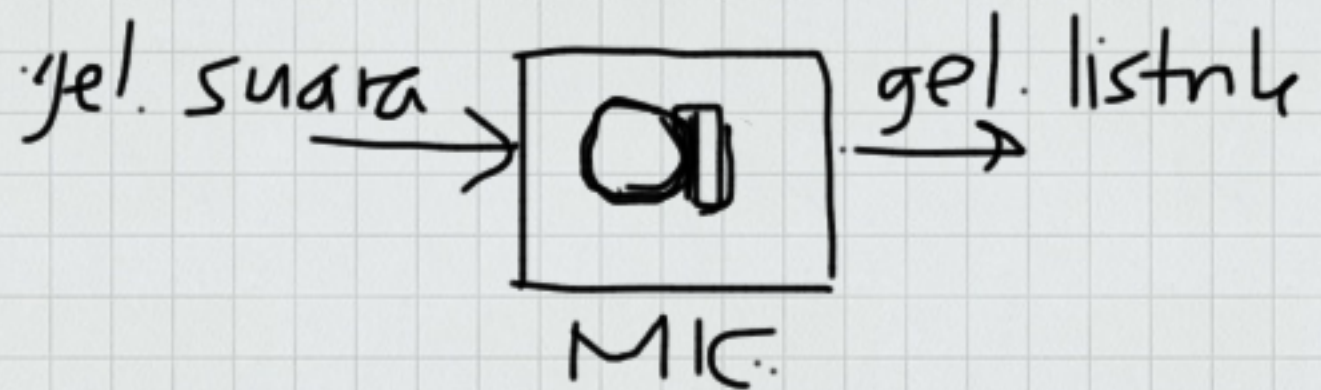
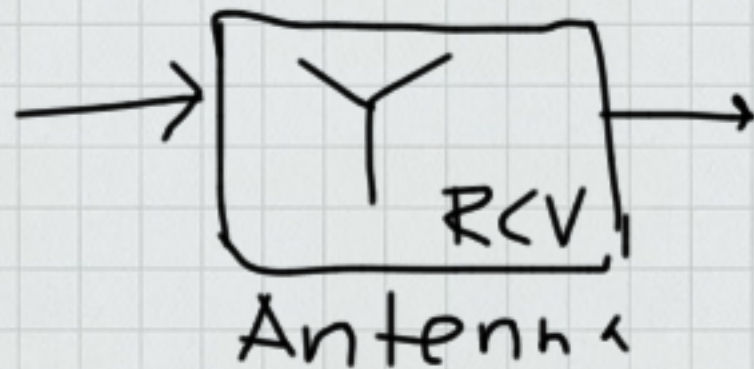
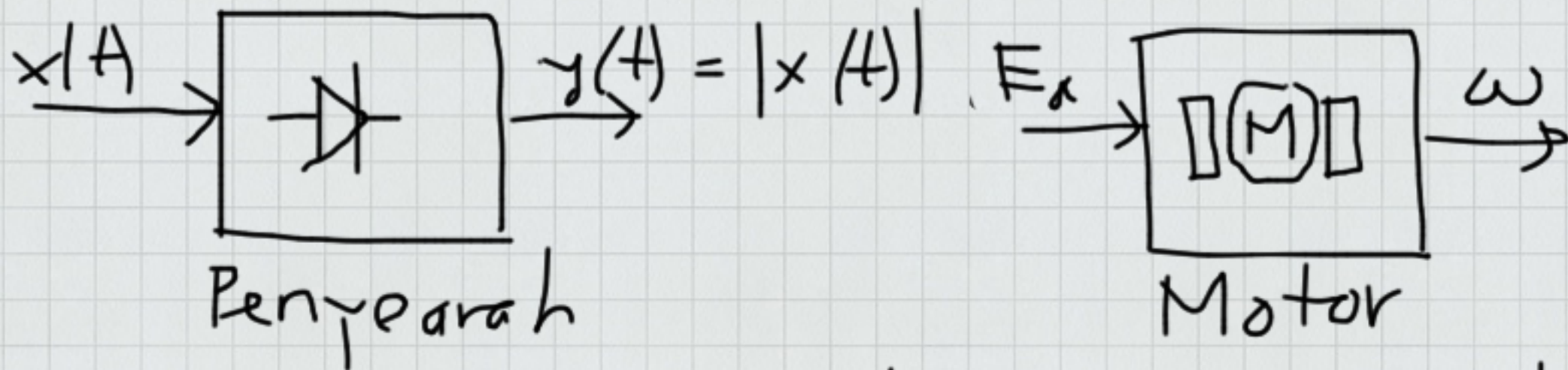


Pengikut Jenuh
(Saturating Amp)

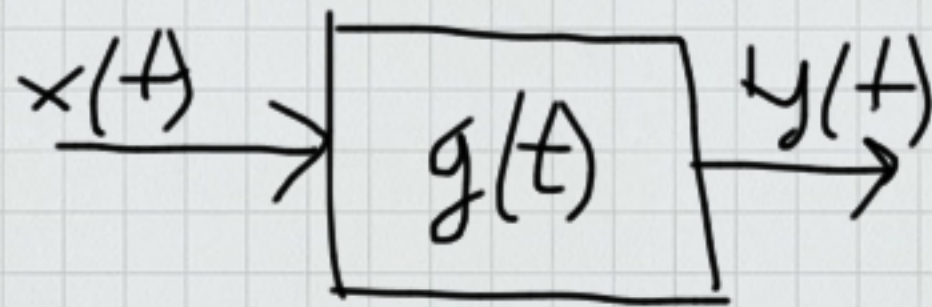


Daerah Mati
(Dead zone)

* Simbol-simbol Khusus

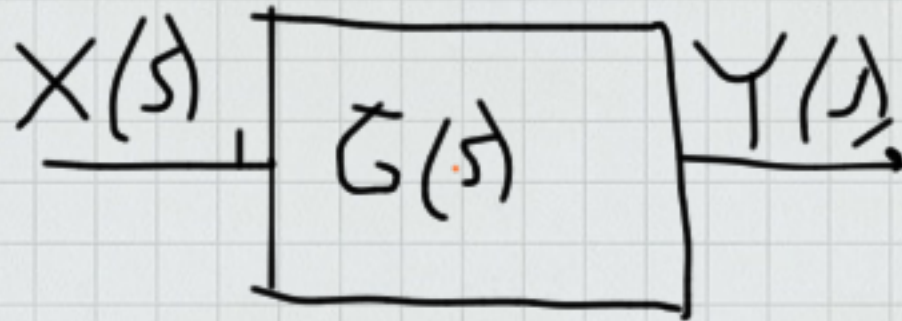


* Nisbah Alih (Transfer Function)



$$~~y(t) = g(t) \cdot x(t)~~$$

tidak
sahih
(valid)

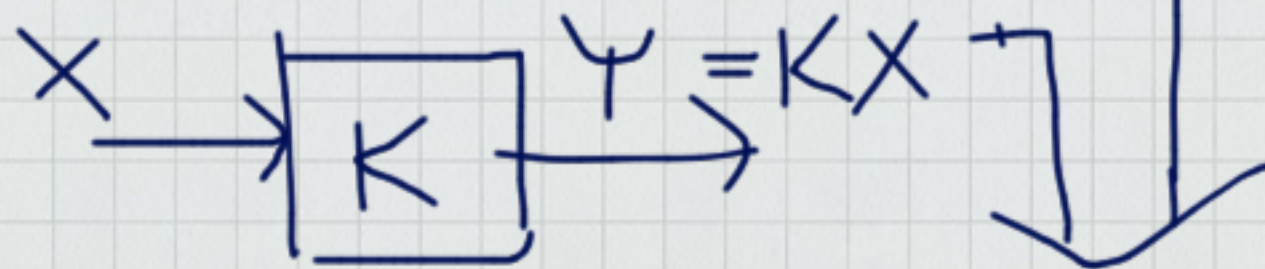
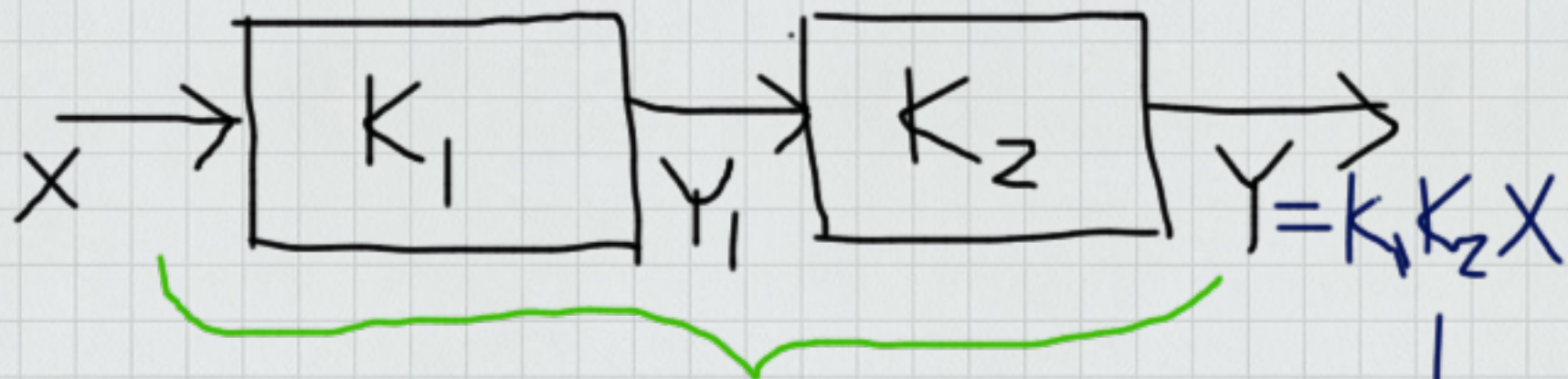


$$Y(s) = G(s) \cdot X(s) \quad \checkmark$$

next = ALJABAR
BAGIAN KUTAK

* ALJABAR BAGAN KOTAK (Block Diagram Algebra)

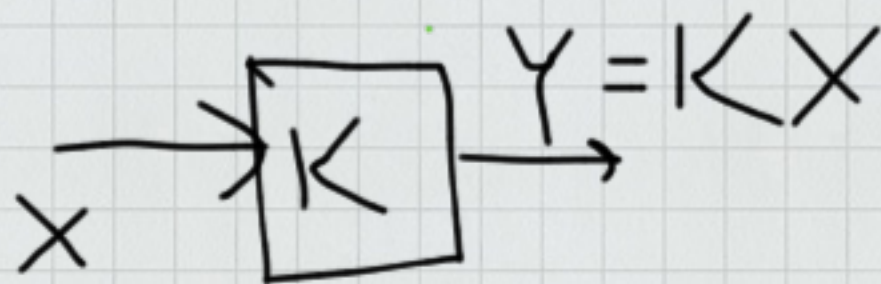
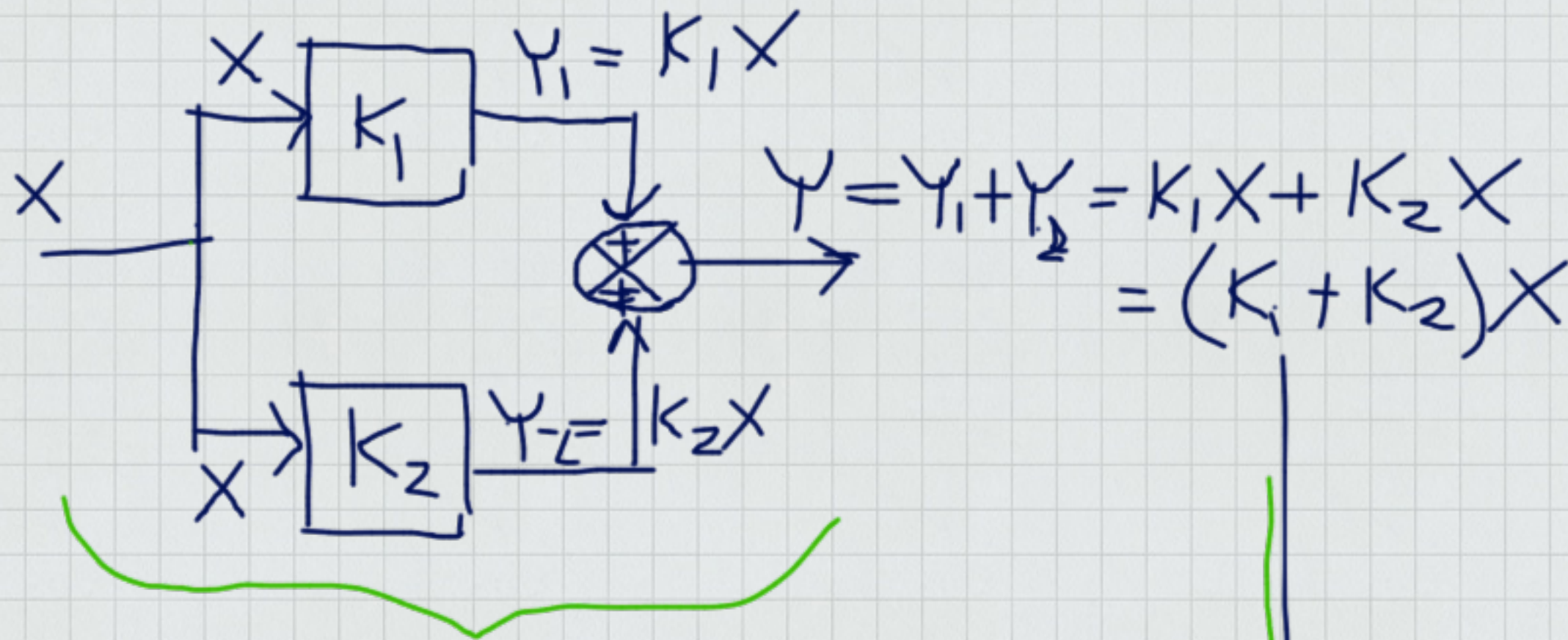
⇒ Hubungan serial (cascade)



$$K = K_1 K_2$$

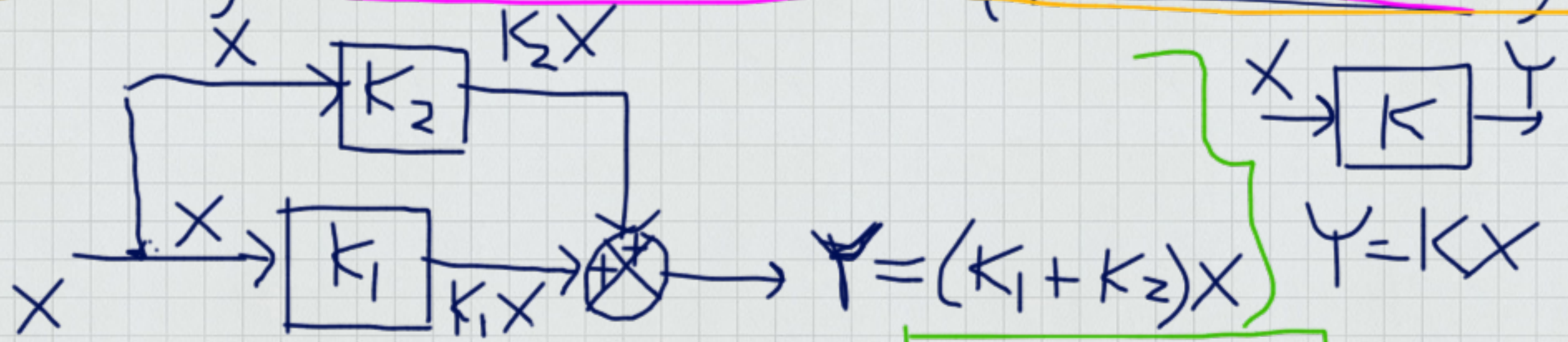
$$\begin{aligned} Y_1 &= K_1 X \\ Y &= K_2 Y_1 \\ &= K_2 (K_1 X) \\ &= K_2 K_1 X \\ &= K_1 K_2 X \end{aligned}$$

⇒ Hubungan PARALEL



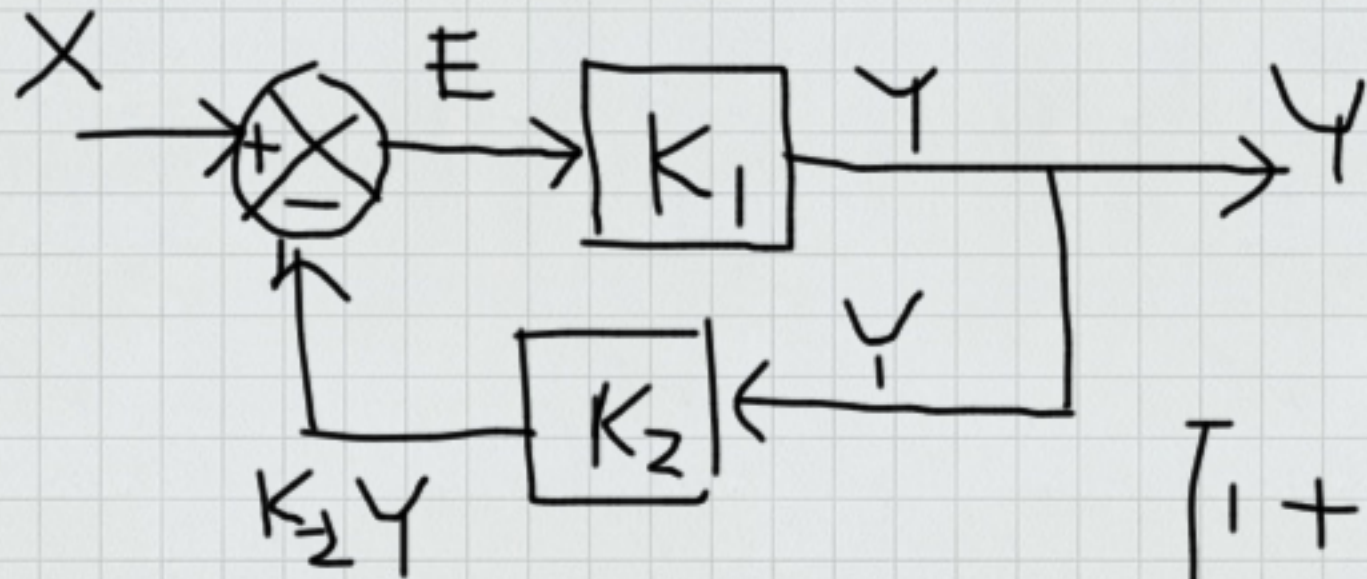
$$K = K_1 + K_2$$

⇒ Hubungan UMPAN MAJU (feed forward)



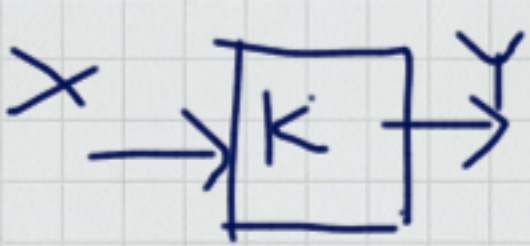
$$K = K_1 + K_2$$

⇒ Hubungan UMPAN BALIK (feedback)



$$\begin{aligned} E &= X - K_2Y \\ Y &= K_1E = K_1(X - K_2Y) \\ &= K_1X - K_1K_2Y \end{aligned}$$

$$[1 + K_1K_2]Y = K_1X \longrightarrow$$



Untuk hubungan Umpan Balik
Negatif (negative feedback)

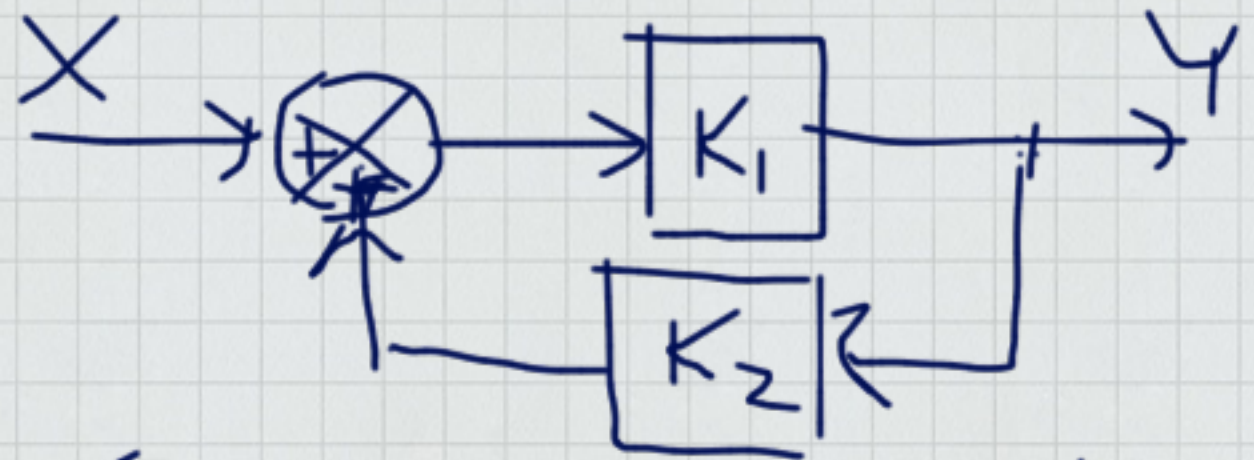
$$Y = KX$$

$$Y = \frac{K_1}{1 + K_1 K_2} X$$

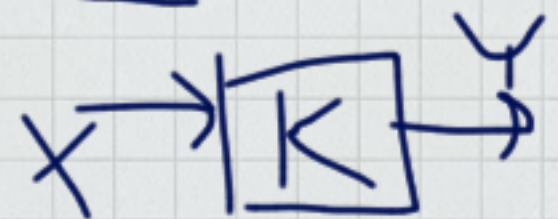
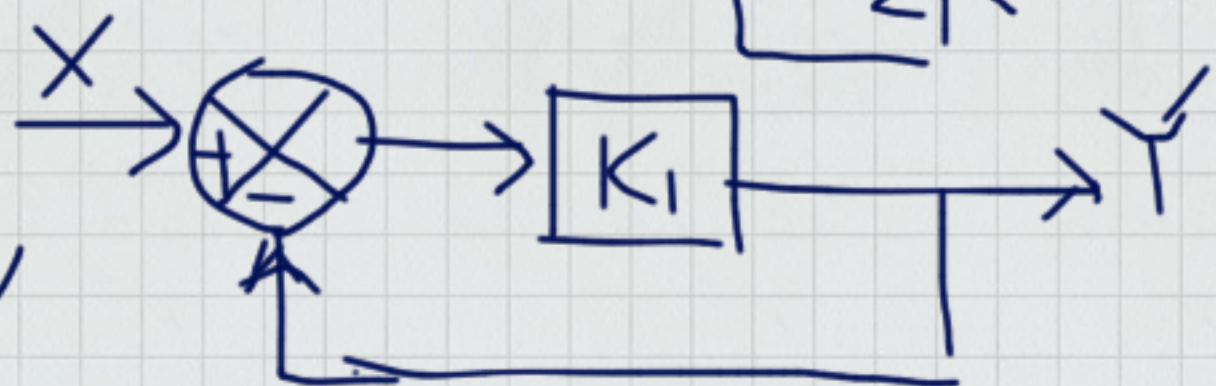
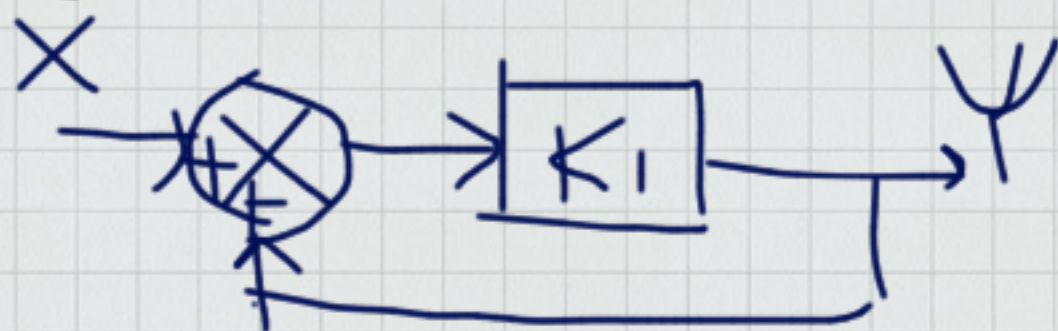
$$K = \frac{K_1}{1 + K_1 K_2}$$

Untuk latihan:

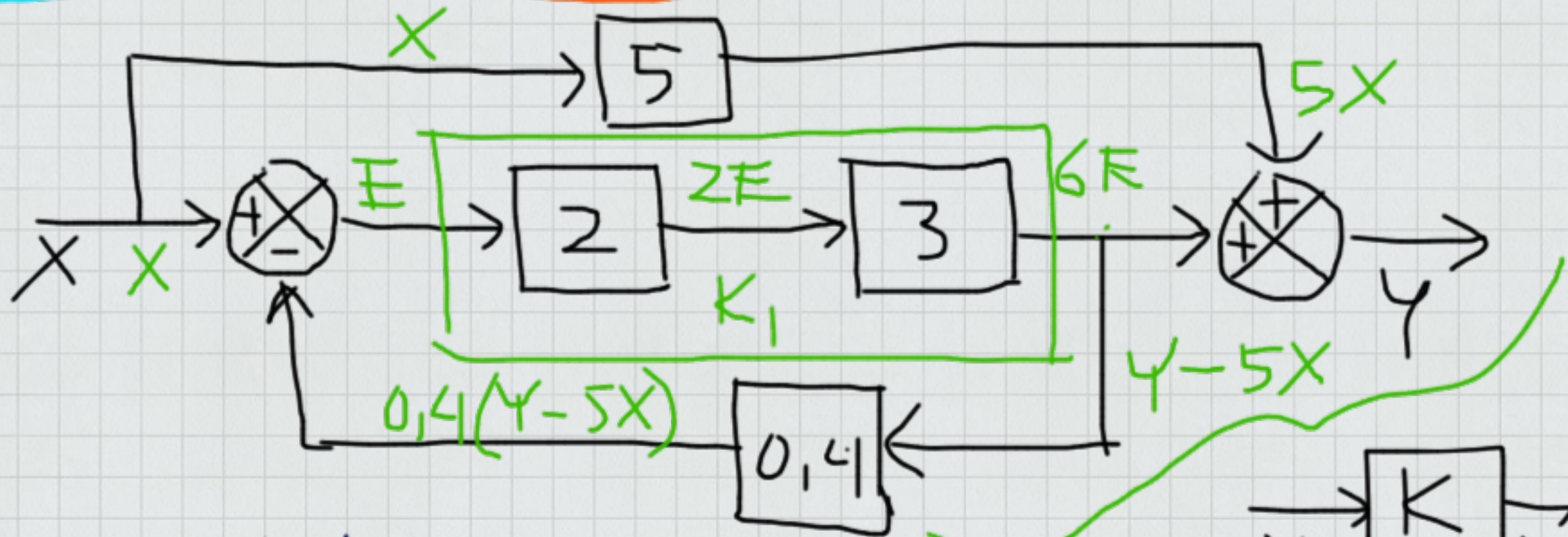
* Umpan Balik Positif:



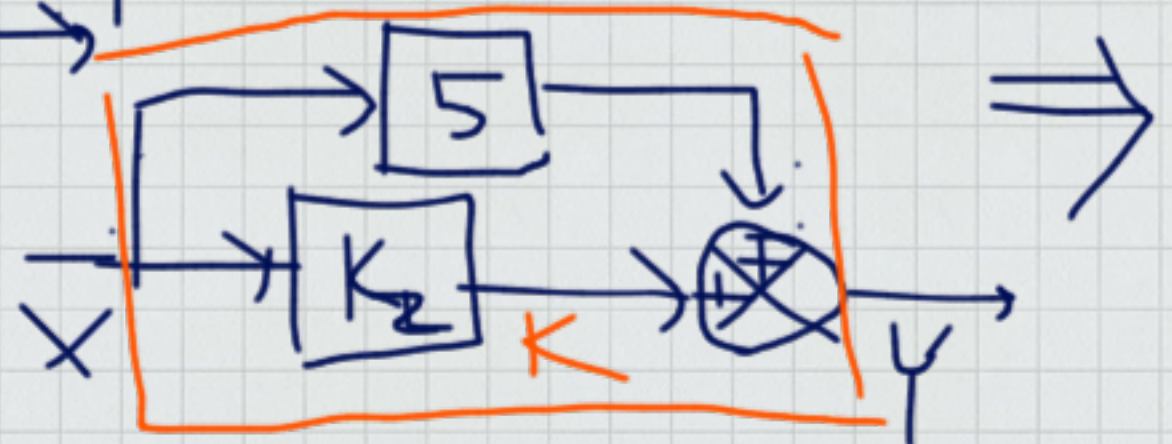
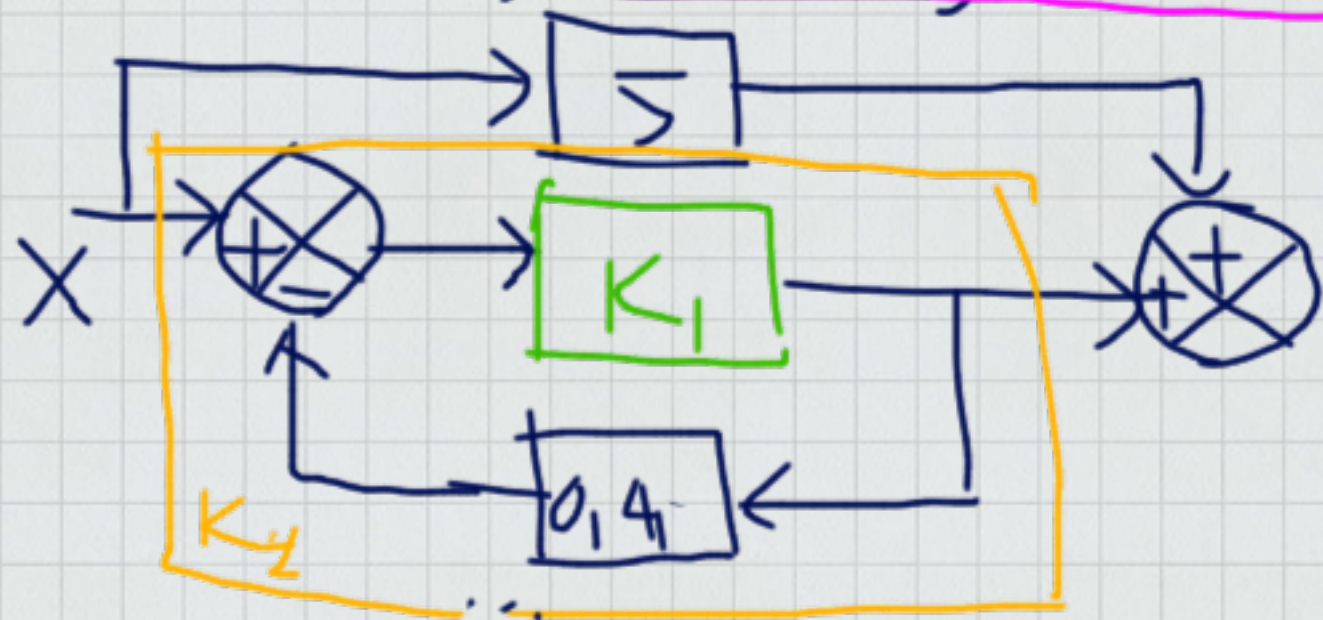
* Umpan Balik Satuan
(Unity feedback)



* Contoh MATEMATIS



Cara I: Aljabar Bagan Kotak



$$K_1 = 2 * 3 = 6$$

$$K = \dots$$

$$K_2 = \frac{K_1}{1 + 0,4K_1} = \frac{6}{1 + 2,4} = \frac{60}{34} = 1 \frac{13}{17}$$

$$K = 5 + K_2 = 6 \frac{13}{17}$$

Cara II Percabangan dan Pertemuan Isyarat

$$F = X - 0,4(Y - 5X) = X - 0,4Y + 2X = 3X - 0,4Y$$

$$6F = Y - 5X$$

$$6(3X - 0,4Y) = Y - 5X$$

$$18X - 2,4Y = Y - 5X$$

$$3,4Y = 23X$$

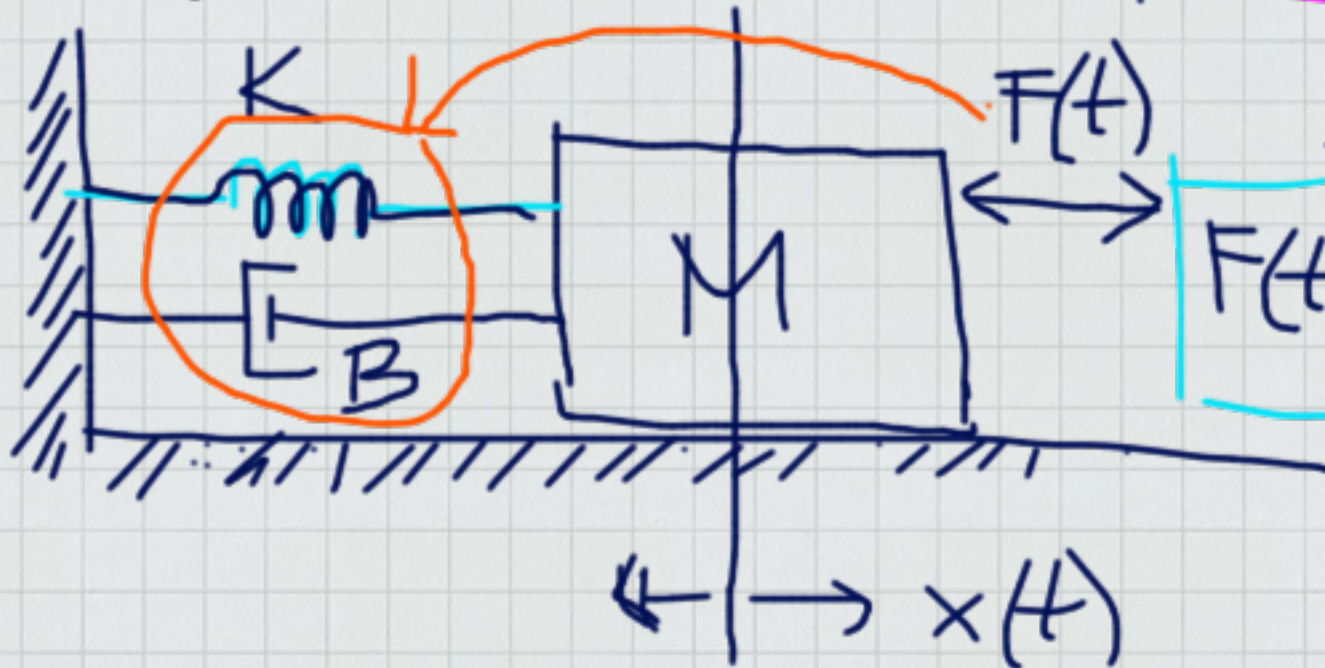
$$Y = \frac{23}{3,4} X = \frac{115}{17} X = 6 \frac{13}{17} X$$

* Contoh MEKANIK

Sistem Suspensi



Spring-Mass-Damper System



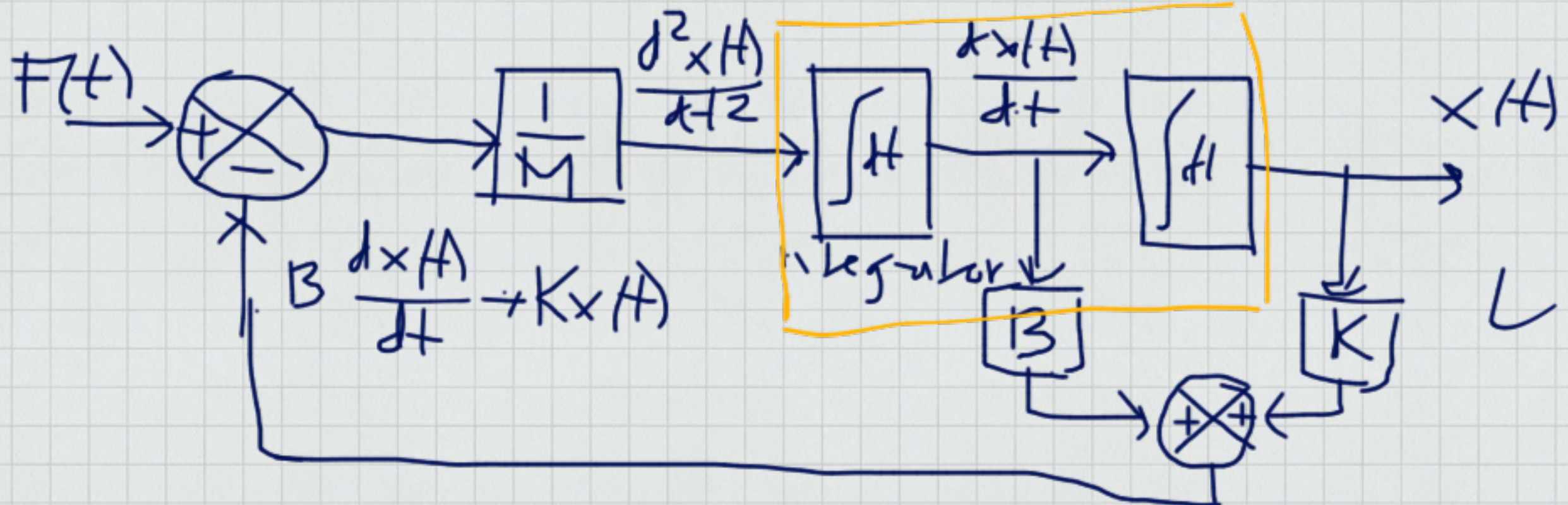
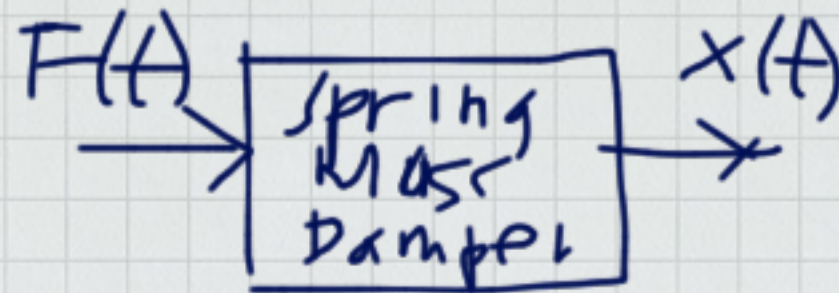
Hukum Newton

$$F(t) = M \frac{d^2 x(t)}{dt^2} + B \frac{dx(t)}{dt} + Kx(t)$$

↑ ↑ ↑
percepatan kecepatan posisi

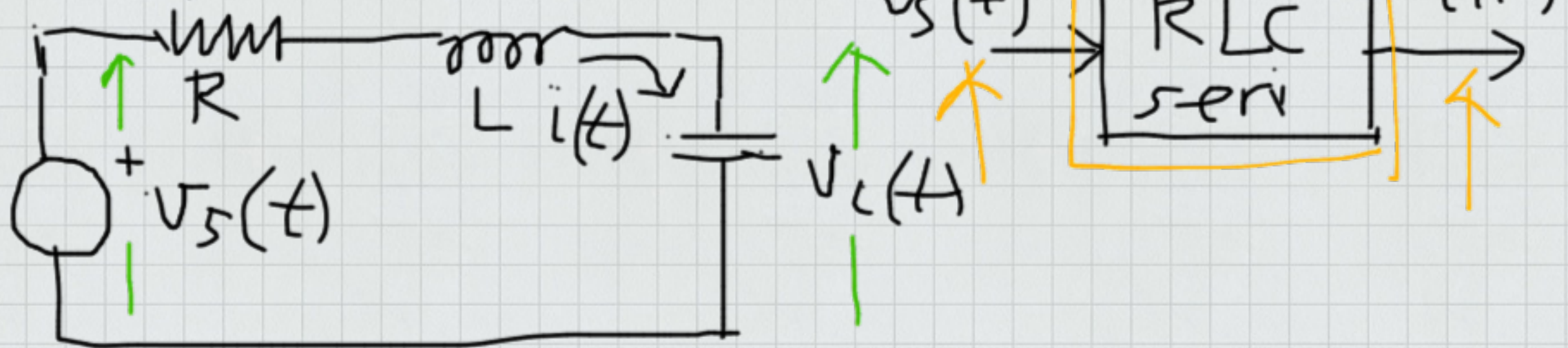
$$\underbrace{\frac{d^2x(t)}{dt^2}}_{\text{perubahan kecepatan}} = \frac{1}{M} \left[\underbrace{F(t)}_{\text{gaya}} - \underbrace{\left(B \frac{dx(t)}{dt} + Kx(t) \right)}_{\text{pegas + shockbreaker}} \right] \quad \checkmark$$

Bagan Kotak



* Contoh ELEKTRIK

Rangkaian Pemuatan Kapasitor
melalui R-L seri

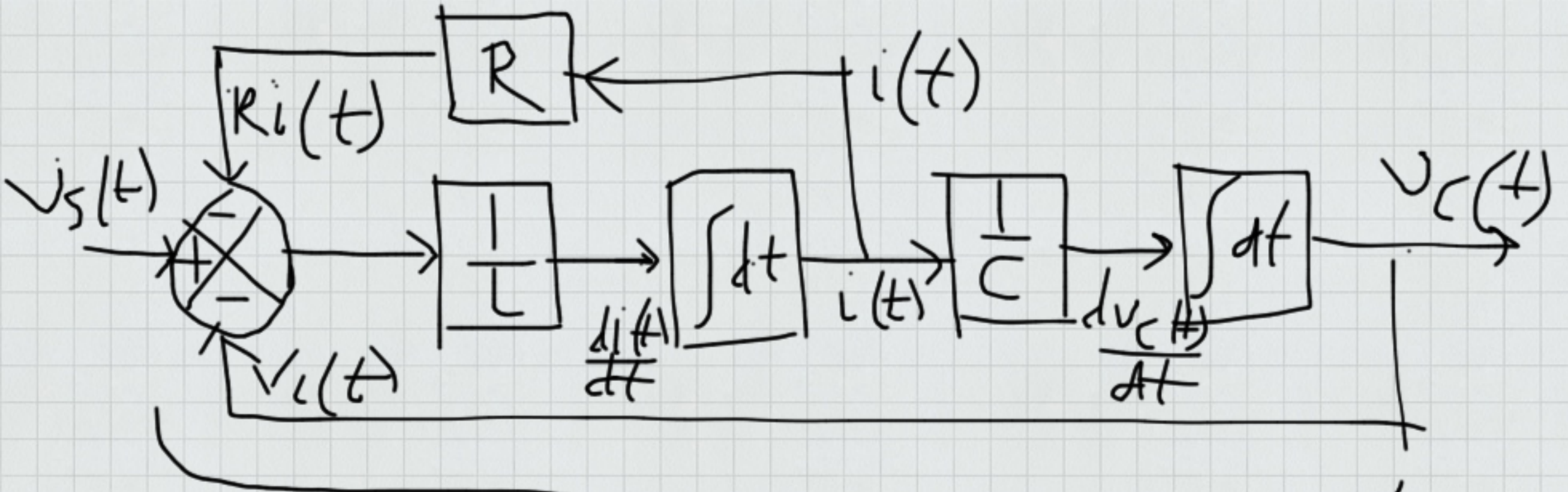


Hukum Ohm:

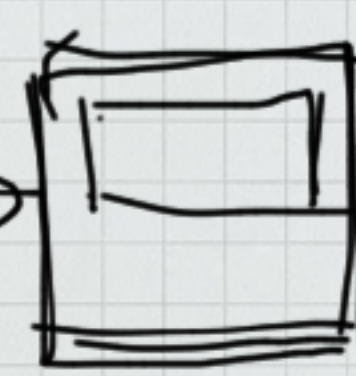
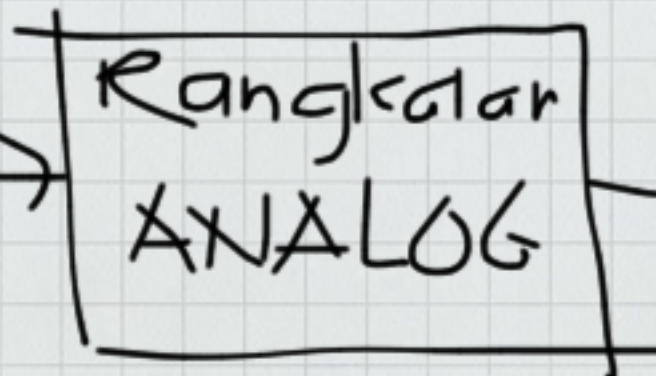
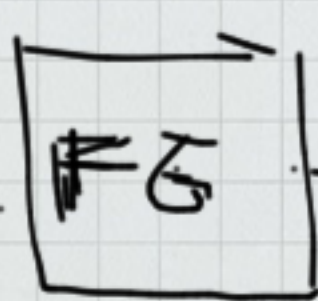
$$(1) \quad V_S(t) - V_C(t) = R i(t) + L \frac{di(t)}{dt}$$

$$(2) \quad i(t) = C \frac{dV_C(t)}{dt} \rightarrow \frac{dV_C(t)}{dt} = \frac{1}{C} i(t)$$

$$(1) \rightarrow \frac{di(t)}{dt} = \frac{1}{L} (V_S(t) - V_C(t) - R i(t))$$



Pembang.
Kit Gelum.
bang



oscilloscope

* Simulasi Analog

* Program Numerik + GUI $\rightarrow$ simulasi Numerik.

BAGAN $\rightarrow$ UNIVERSAL
KOTAK

* MODEL NISBAH ALIH dan Transf. LAPLACE

Transfer Function

Alih

Fungsi

Jelmaan

Δ : DELTA

δ : delta

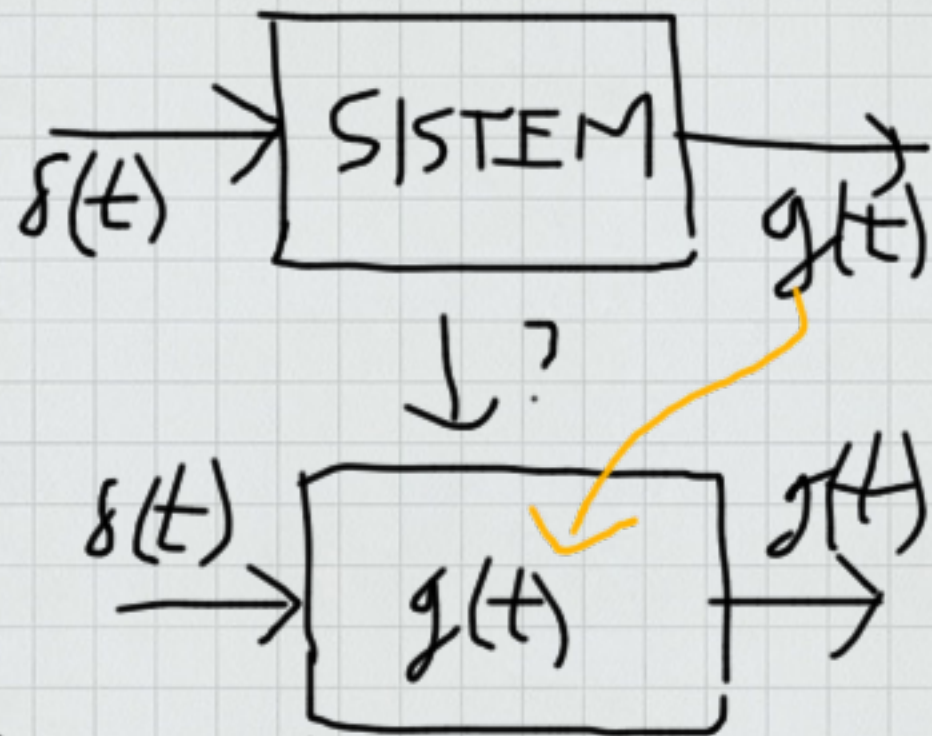
⇒ Apa itu Nisbah Alih ???

What is a Transfer Function.

Jawaban singkat : Nisbah Alih adalah
Tanggapan Denyut

A Transfer Function is the
Impulse Response

Penjelasan: Jika suatu sistem diberi masukan
isyarat denyut satuan (unit impulse, $\delta(t)$),
maka keluarannya adalah Nisbah Alih
sistem tsb



* "Keras" dan "cepat"

* Karakteristik $\delta(t)$:

$\Rightarrow$ Hanya ada pada $t=0$

$$\delta(t) \begin{cases} = 0, & t \neq 0 \\ \neq 0, & t = 0 \end{cases}$$

$\Rightarrow$ Luas bidang antara $\delta(t)$ dan sumbu adalah satu satuan luas.

$\Rightarrow$ Apa itu syarat denyut satuan?
(unit impulse, $\delta(t)$)

* Tidak bisa direalisasi

* Mirip $\delta(t)$:

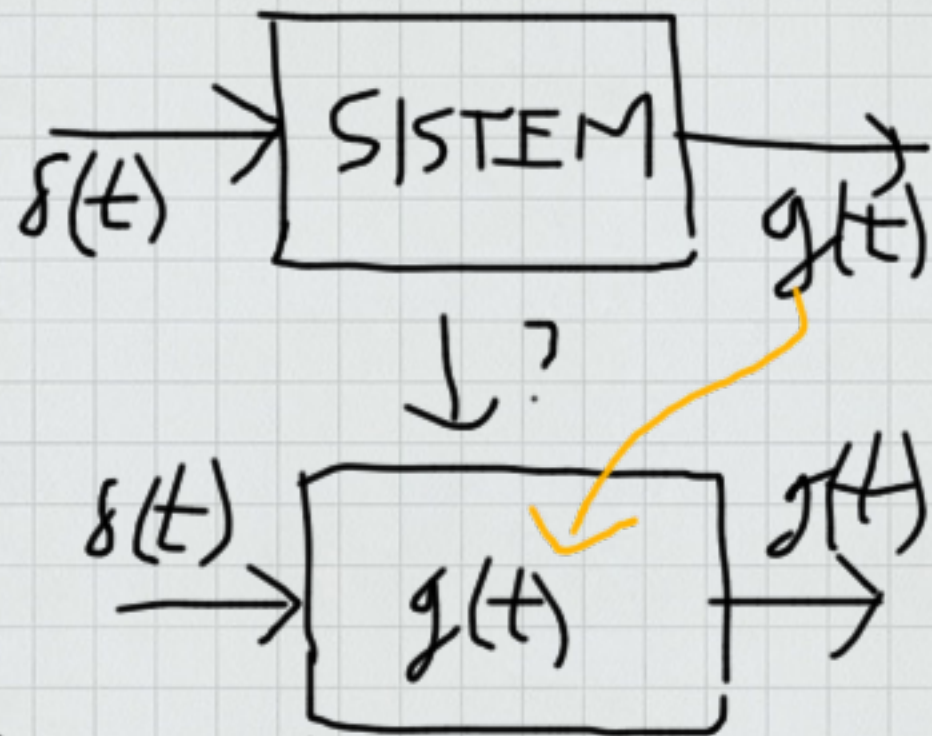
$\rightarrow$ Sambaran petir

$\rightarrow$ pukulan stick golf pada bola golf.

$\rightarrow$ percikan bunga api

$t = \text{time / waktu}$

$\delta(t)$ dan sumbu adalah $\int \delta(t) dt = 1$



* "Keras" dan "cepat"

* Karakteristik $\delta(t)$:

⇒ Hanya ada pada $t=0$

$$\delta(t) \begin{cases} = 0, & t \neq 0 \\ \neq 0, & t = 0 \end{cases}$$

⇒ Luas bidang antara $\delta(t)$ dan sumbu adalah satu satuan luas.

⇒ Apa itu syarat denyut satuan?
(unit impulse, $\delta(t)$)

* Tidak bisa direalisasi

* Mirip $\delta(t)$:

→ Sambaran petir

→ pukulan stick golf pada bola golf.

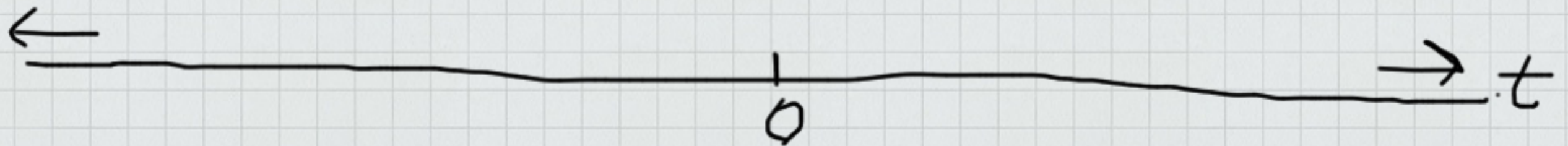
→ percikan bunga api

$t = \text{time / waktu}$

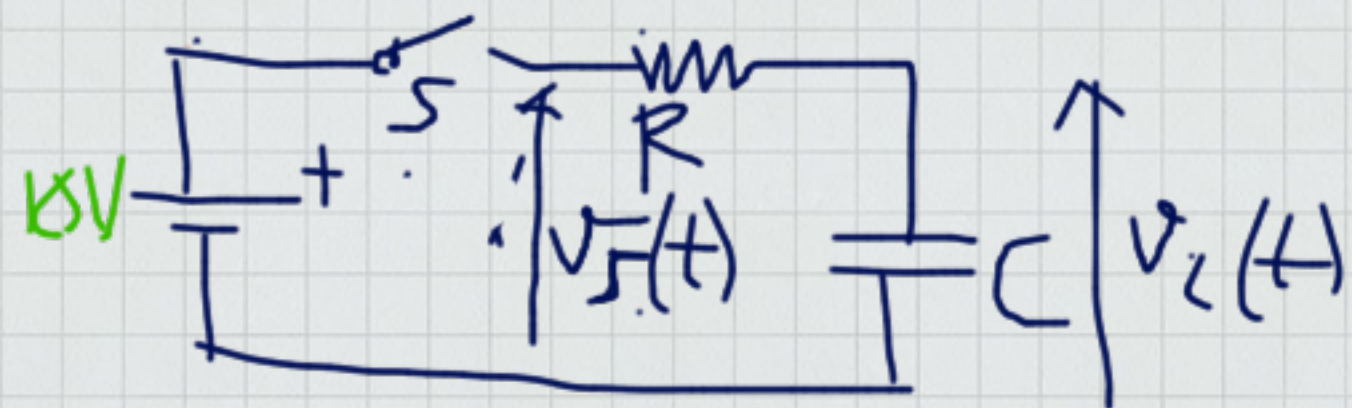
$\delta(t)$ dan sumbu adalah $\int \delta(t) dt = 1$

* Bagaimana membuat $\delta(t)$???

Ada banyak sekali cara ————



* Contoh Elektrik



$t \leq 0$, C tidak bermuatan

* Aljabar bagan kotak

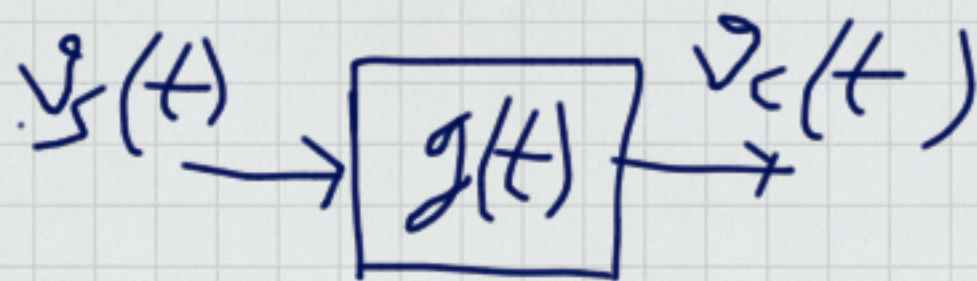
$$G(s) = \frac{V_c(s)}{V_s(s)}$$

* Transf. Laplace.

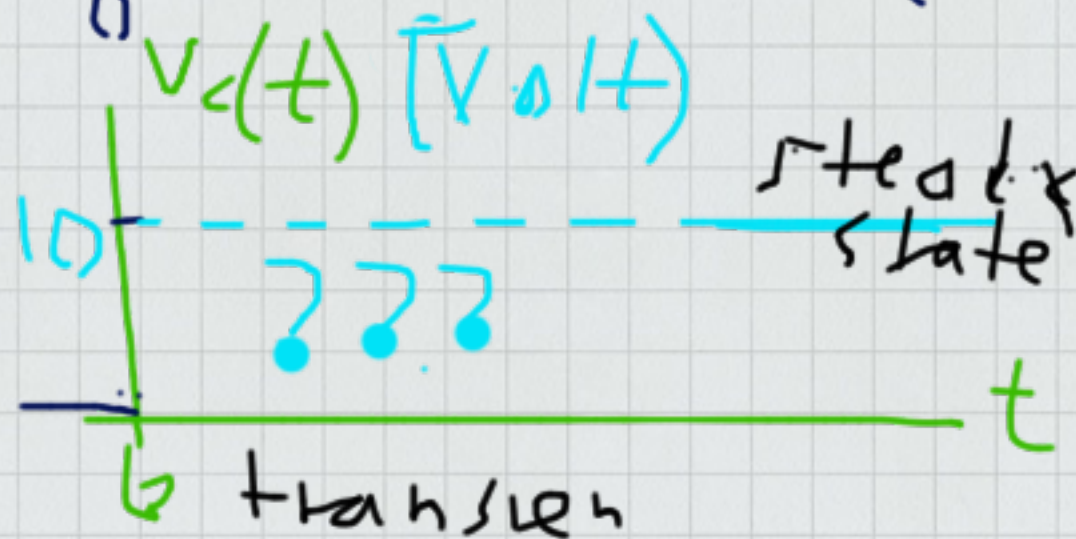
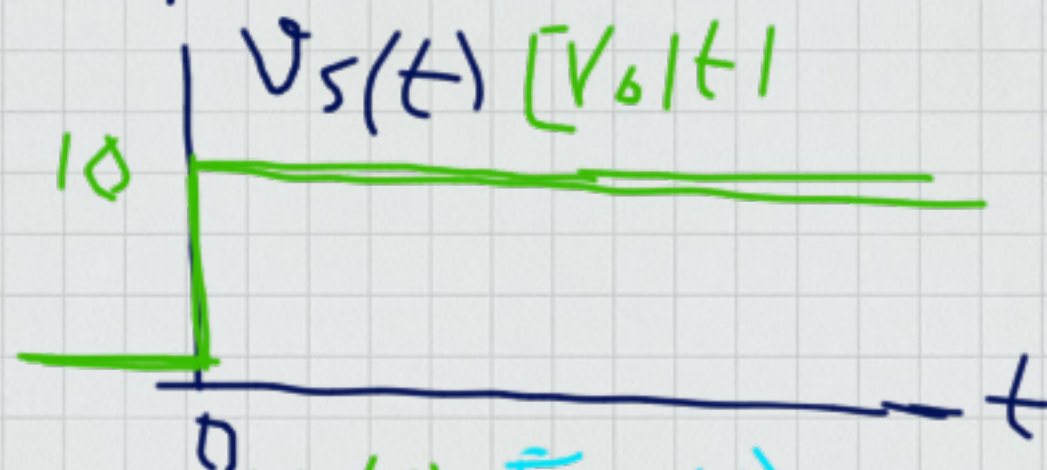
* $V_s(s) =$

* $V_c(s) =$

* $V_c(t) = \mathcal{L}^{-1} V_c(s)$



Jika saklar S ditutup pada $t = 0$



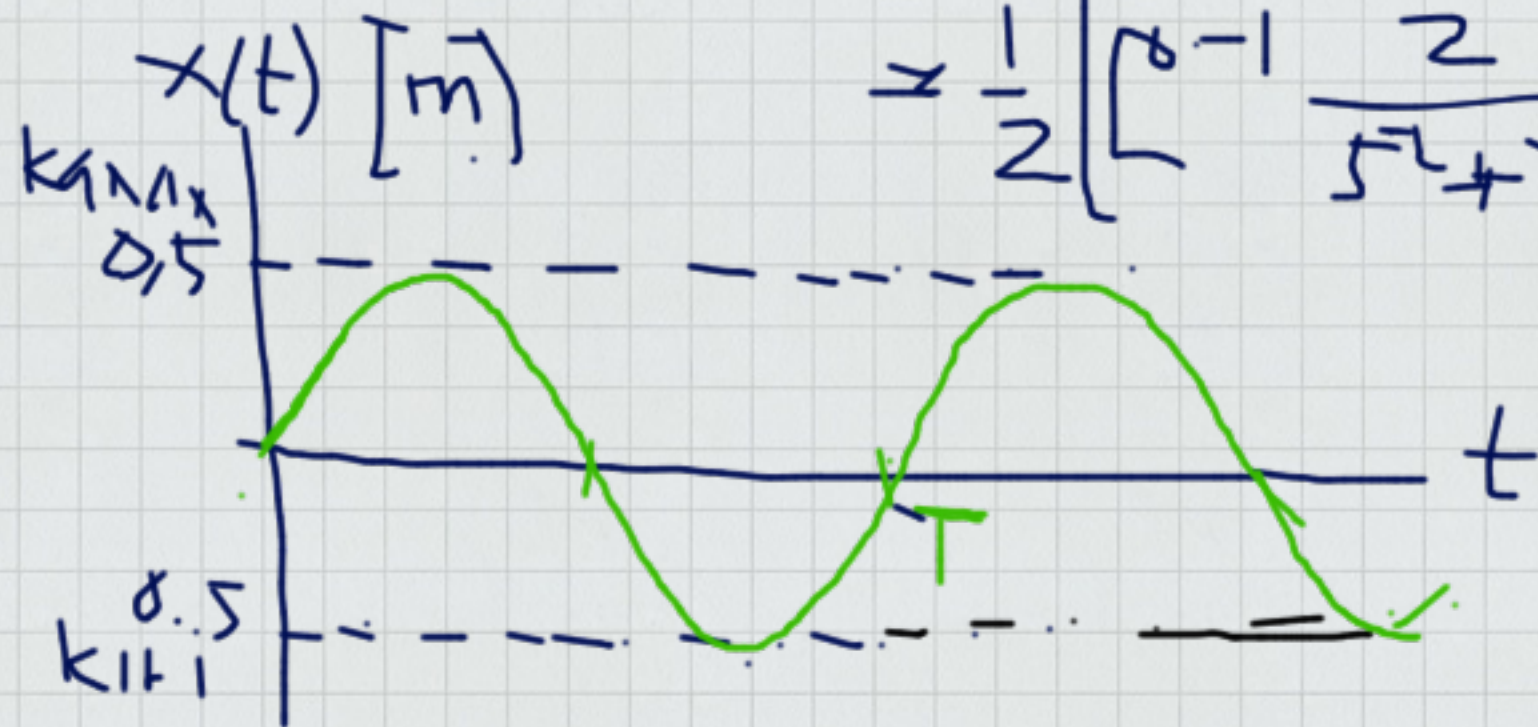
$$Ms^2 X(s) + KX(s) = F(s)$$

$$[Ms^2 + K]X(s) = F(s) \rightarrow G(s) = \frac{X(s)}{F(s)} = \frac{1}{Ms^2 + K}$$

$$= \frac{1}{s^2 + 4}$$

$$x(t) = \mathcal{L}^{-1} G(s) = \mathcal{L}^{-1} \frac{1}{s^2 + 4} = \mathcal{L}^{-1} \frac{1}{2} \left[\frac{2}{s^2 + 2^2} \right]$$

$$= \frac{1}{2} \left[\mathcal{L}^{-1} \frac{2}{s^2 + 2^2} \right] = \frac{1}{2} \sin 2t \quad [m]$$

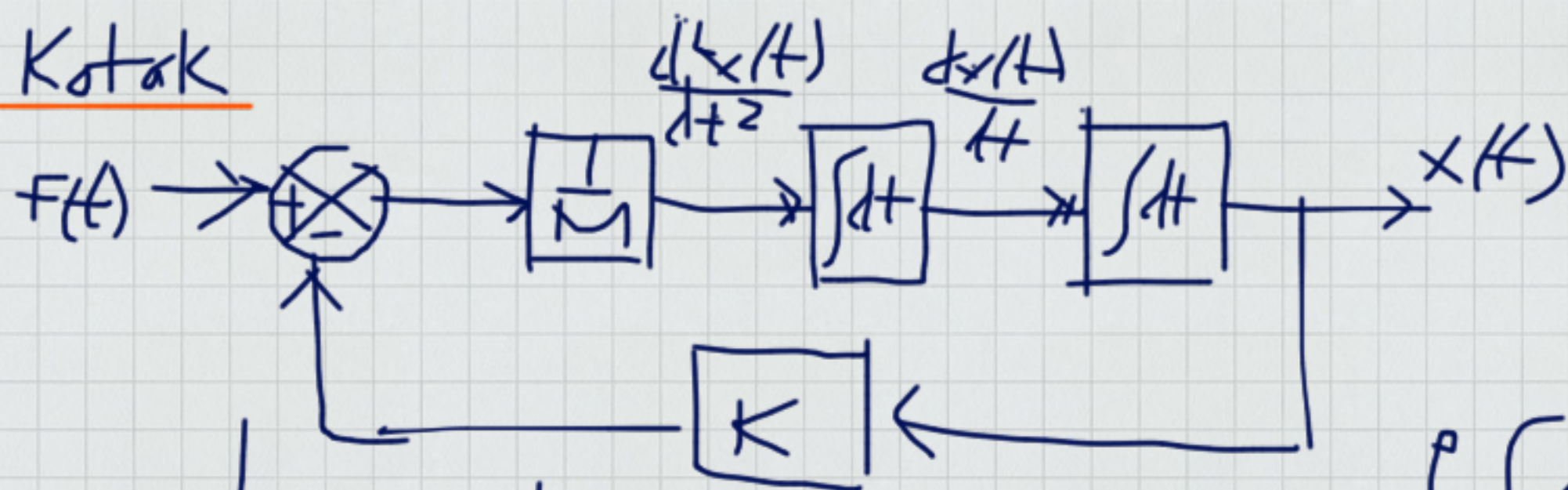


$$T = ? \quad [sec]$$

$$\omega = \frac{2\pi}{T} = 2$$

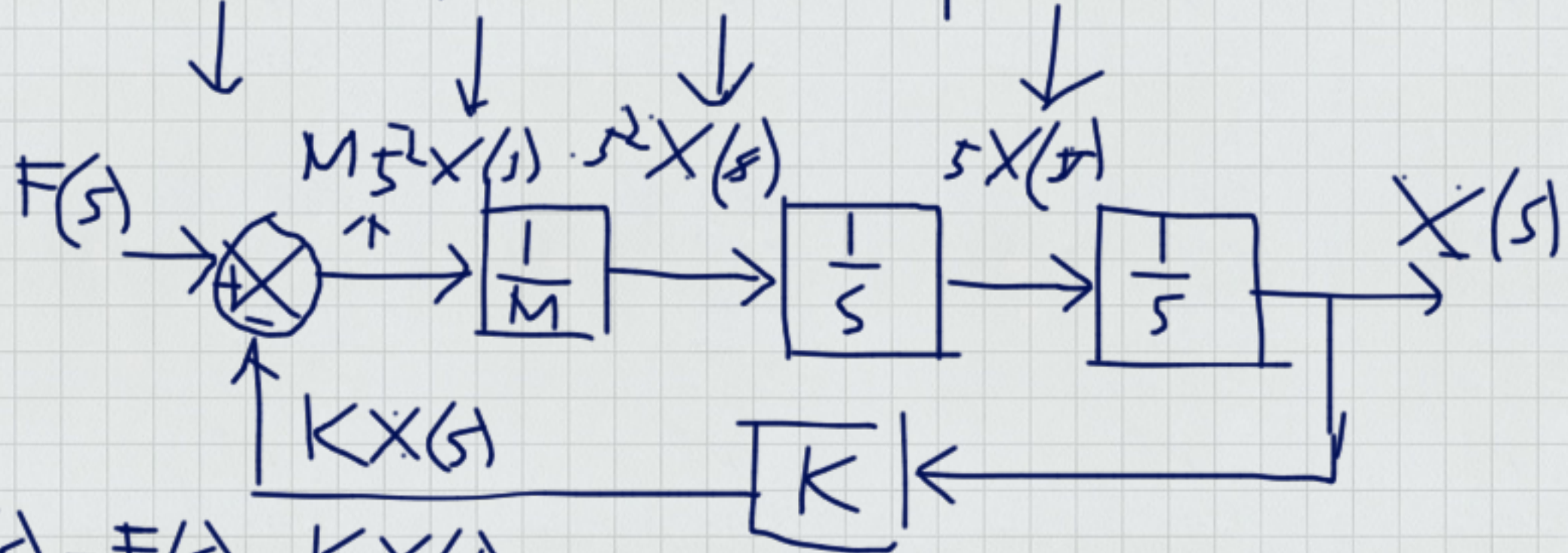
$$2T = 2\pi \rightarrow T = \pi \text{ sec}$$

Bagan Kotak



$$\mathcal{L} \int dt = ?$$

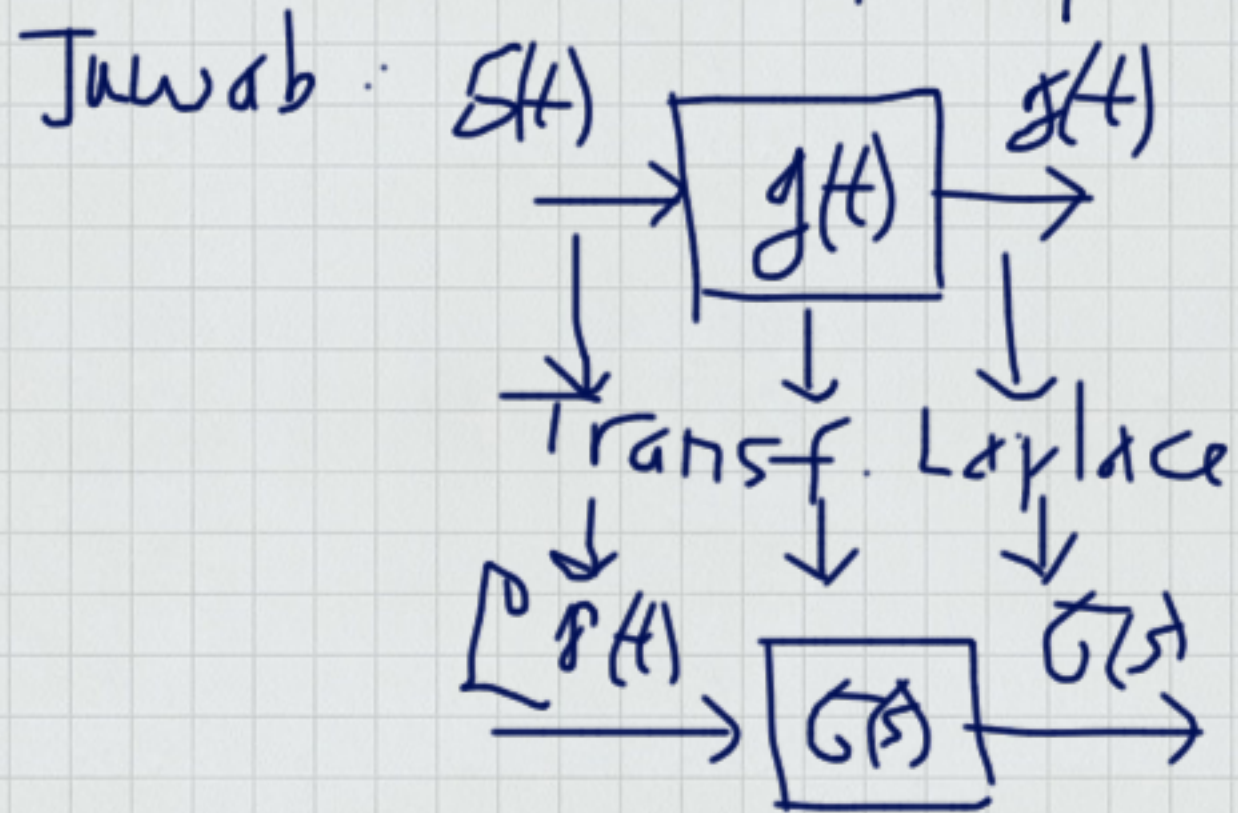
Transformasi Laplace



$$M s^2 X(s) = F(s) - K X(s)$$

* Contoh Matematis

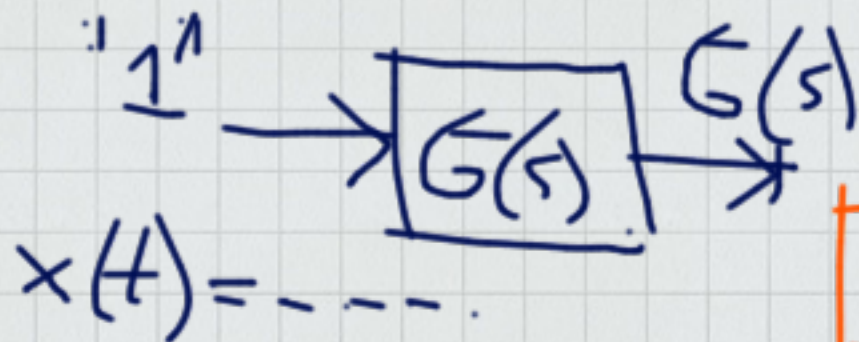
Tentukan Transf. Laplace dari $\delta(t)$! $\mathcal{L}\{\delta(t)\} = ?$



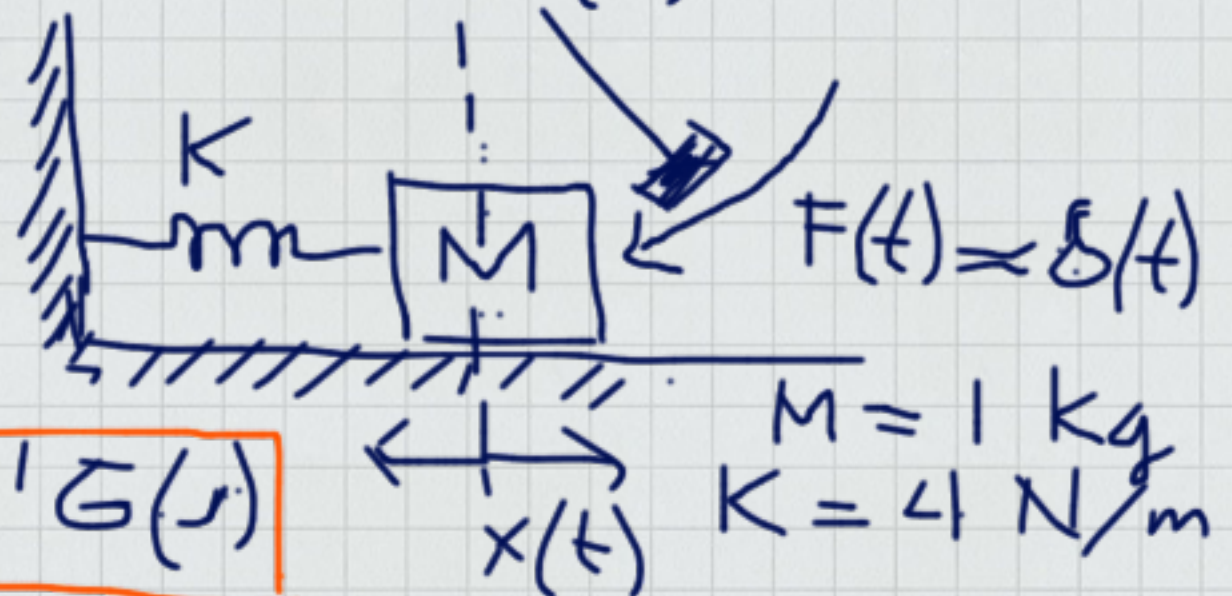
$$\Rightarrow G(s) = G(s) \int_0^\infty \delta(t) dt$$

$$\int_0^\infty \delta(t) dt = \frac{G(s)}{G(s)} = 1$$

* Contoh Mekanik



$$x(t) = \mathcal{L}^{-1}\{G(s)\}$$



Dengan Transformasi Laplace.

"t"



"s"

(peubah REAL)

besaran fisik

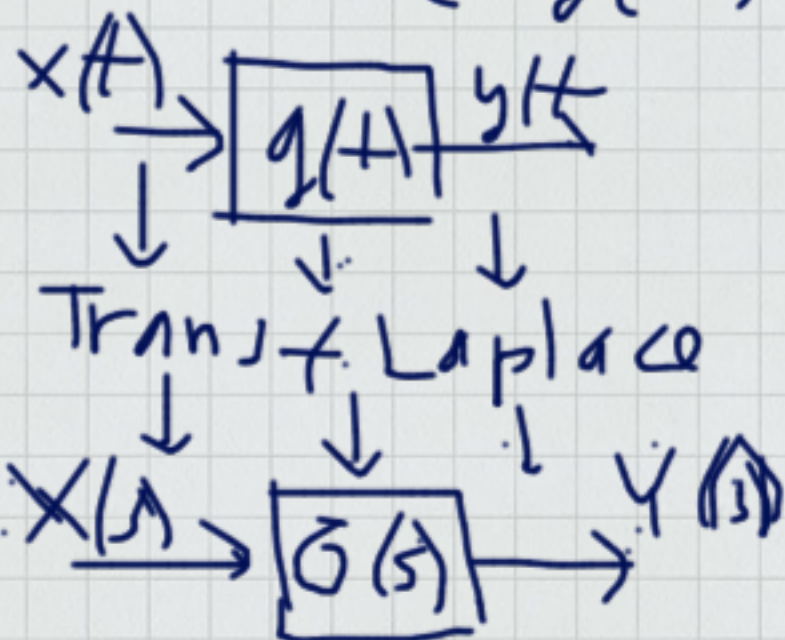
(peubah COMPLEX)

bukan besaran fisik

$$\mathcal{L}\{x(t)\} = X(s) \iff \mathcal{L}^{-1}\{X(s)\} = x(t)$$

$$\mathcal{L}\{g(t)\} = G(s) \iff \mathcal{L}^{-1}\{G(s)\} = g(t)$$

$$\mathcal{L}\{y(t)\} = Y(s) \iff \mathcal{L}^{-1}\{Y(s)\} = y(t)$$



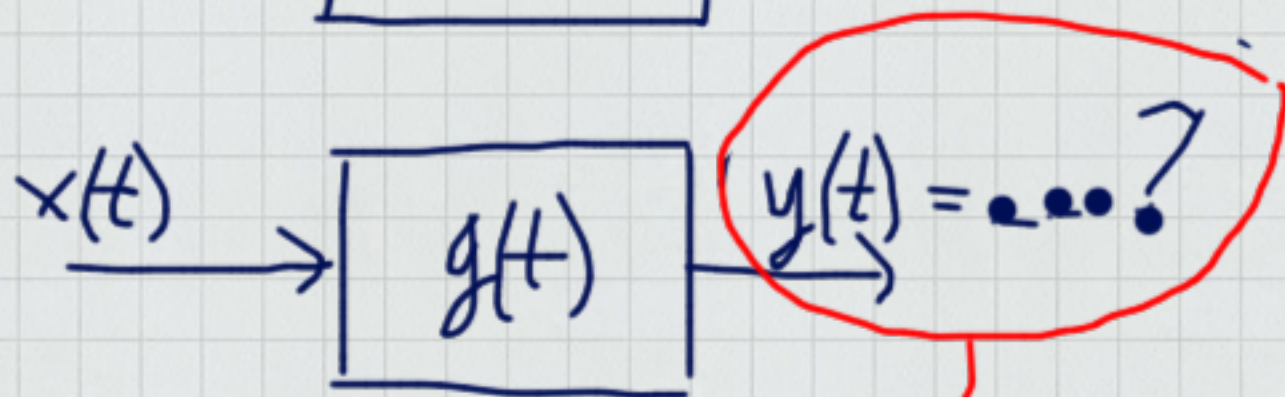
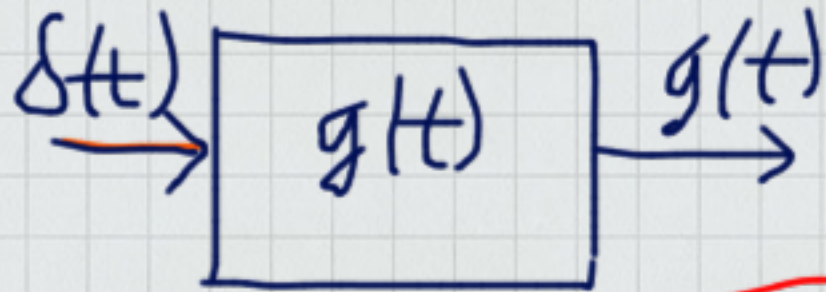
~~$$y(t) = g(t) \cdot x(t)$$~~

TIDAK SAHIT

$$Y(s) = G(s) \cdot X(s)$$

SAHIT

* Transformasi Laplace



$$x(t) \neq \delta(t) \rightarrow y(t) \neq g(t)$$

dengan integral KONVOLUSI:

$$y(t) = \int_{-\infty}^{\infty} g(t-\tau) x(\tau) d\tau$$

$$= \int_{-\infty}^{\infty} g(\tau) x(t-\tau) d\tau$$

~~$$y(t) = g(t) \cdot x(t)$$~~

$g(t) \cdot x(t)$ TIDAK SAH

$y(t)$ adalah KONVOLUSI
antara $g(t)$ dan $x(t)$,

τ = "tau"

"dihindari"
dengan
Transf. Laplace

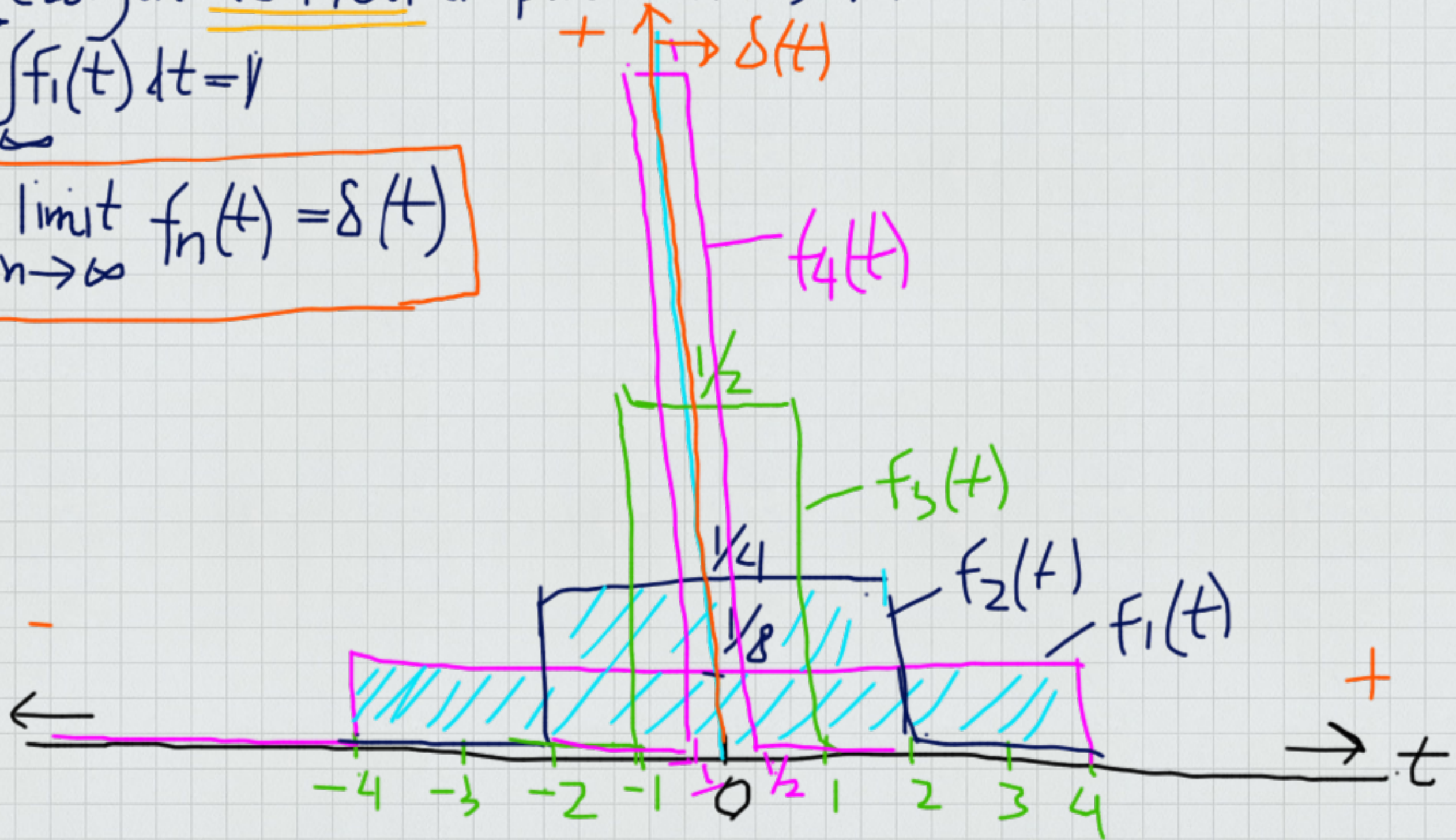
* Bagaimana membuat $\delta(t)$???

Ada banyak sekali cara
Sebagai contoh dapat diilustrasikan sbb :

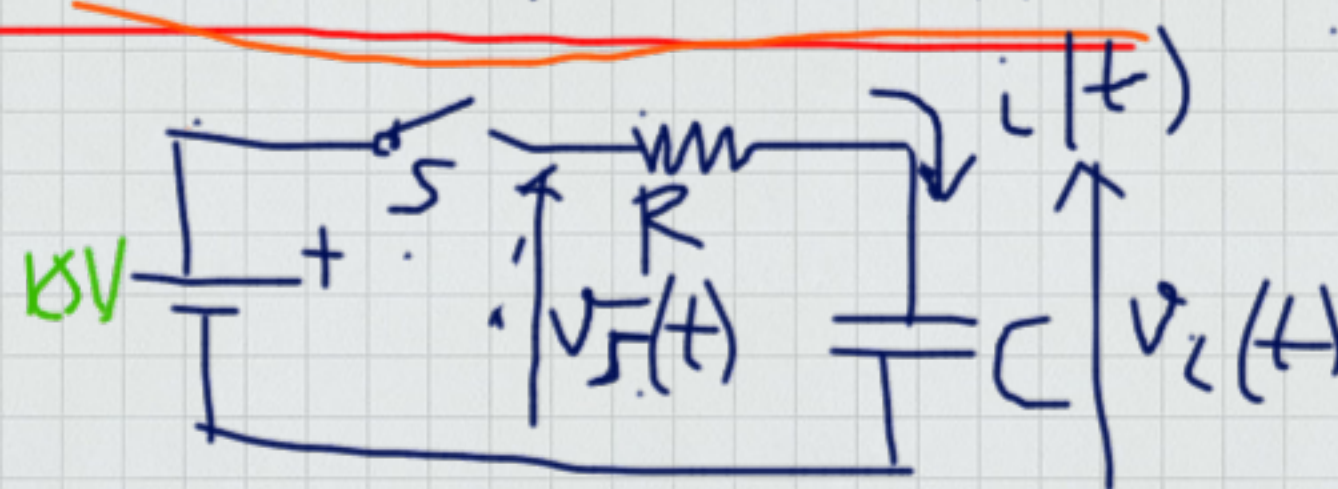
$$\int_{-\infty}^{\infty} f_1(t) dt = 1$$

$$\lim_{n \rightarrow \infty} f_n(t) = \delta(t)$$

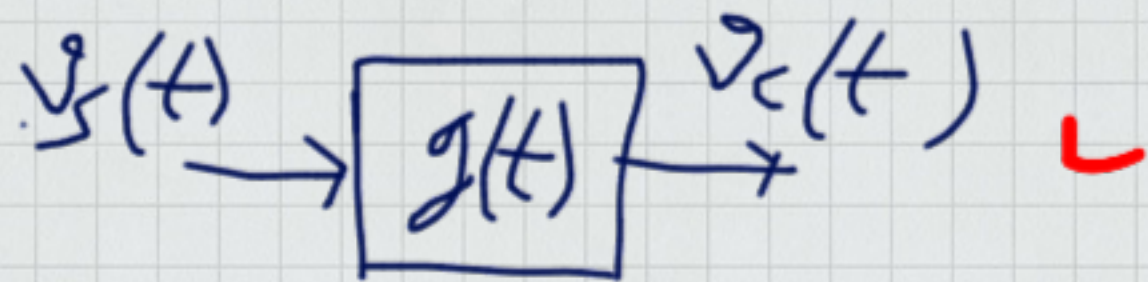
$\delta(t)$



* Contoh Elektrik



$t \leq 0$, C tidak bermuatan



Jika saklar S ditutup pada $t = 0$

* Aljabar bagan kotak

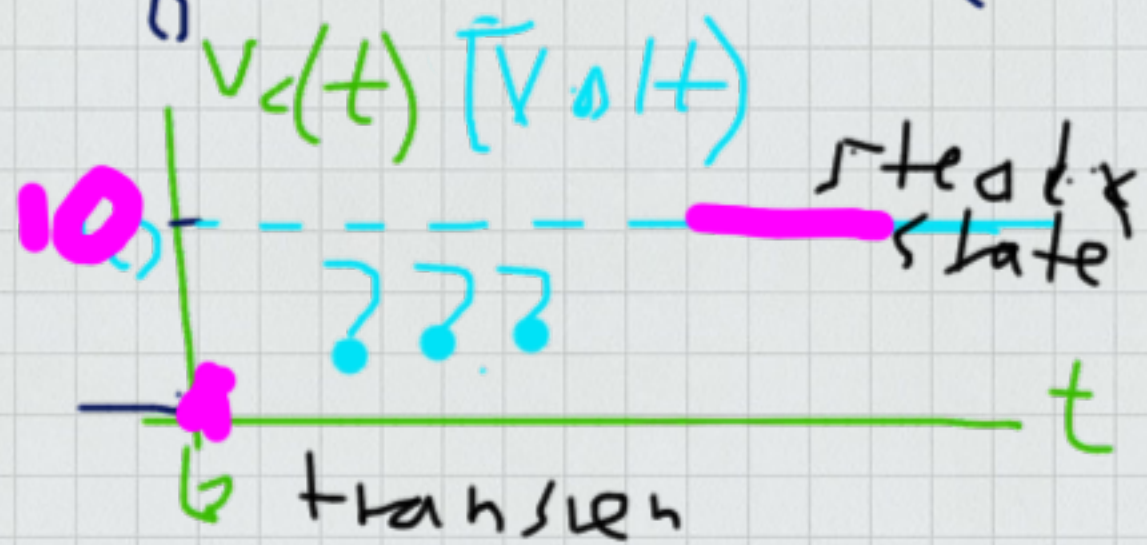
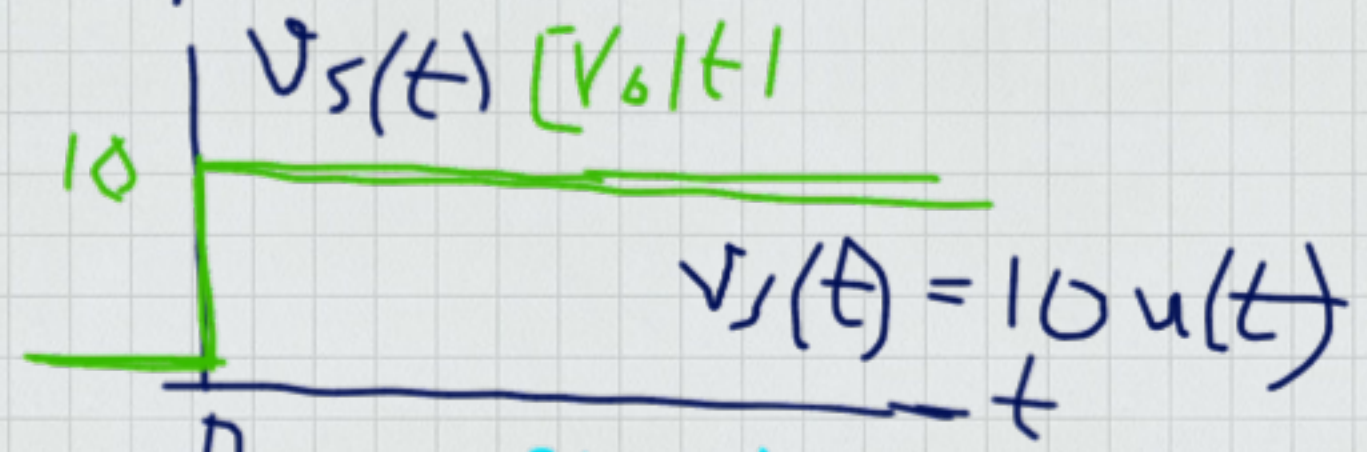
$$G(s) = \frac{V_c(s)}{V_s(s)}$$

* Transf. Laplace

* $V_s(s) = \dots$

* $V_c(s) = \dots$

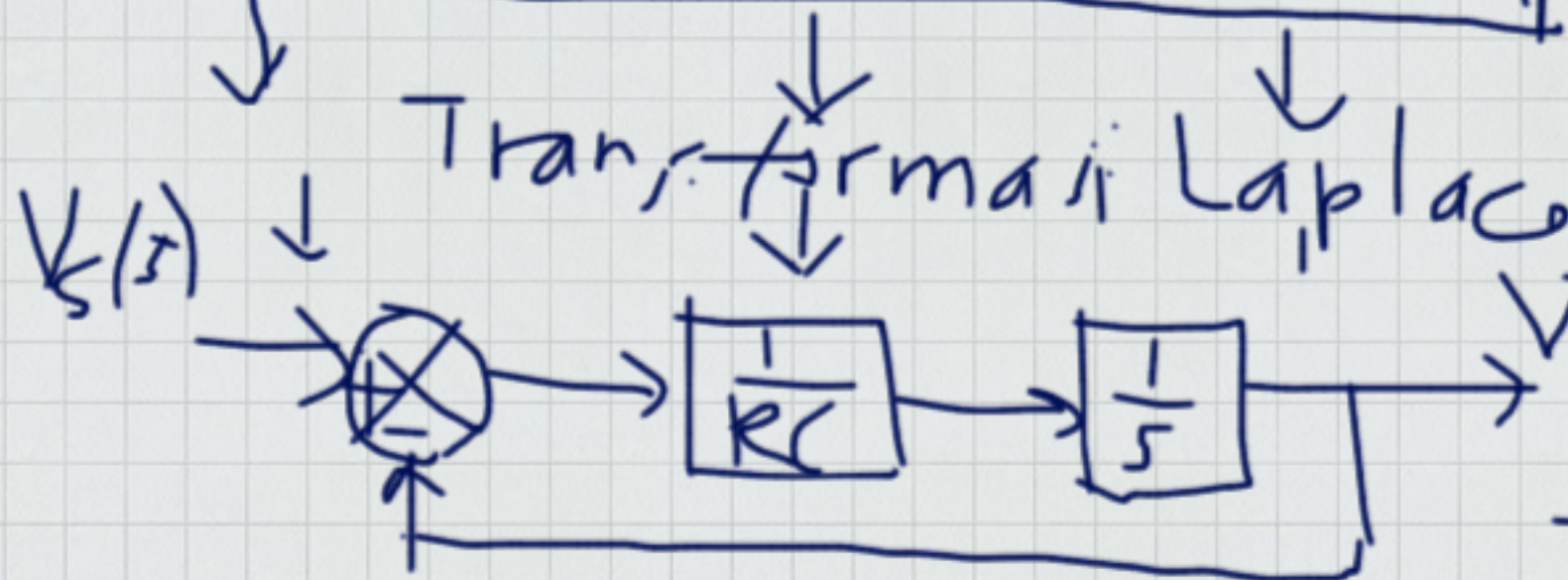
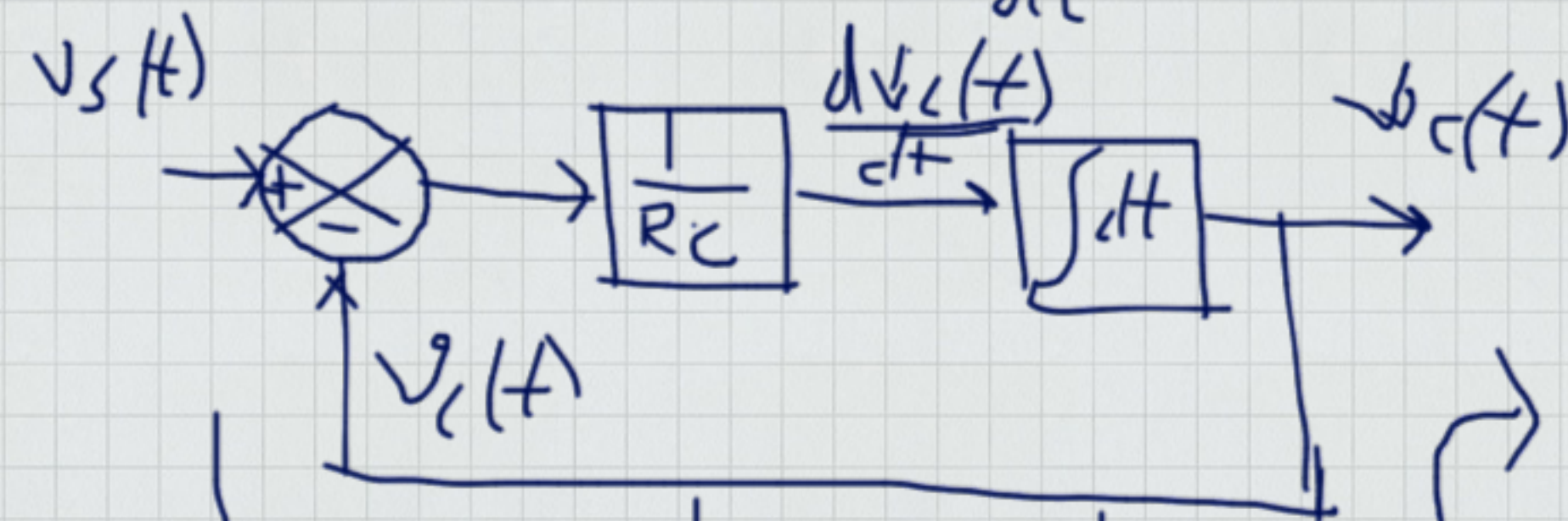
* $V_c(t) = \mathcal{L}^{-1} V_c(s)$



Jawab: * Hukum Ohm.

$$\left. \begin{aligned} i(t) &= \frac{V_s(t) - V_c(t)}{R} \\ i(t) &= C \frac{dV_c(t)}{dt} \end{aligned} \right\}$$

$$\frac{dV_c(t)}{dt} = \frac{V_s(t) - V_c(t)}{RC}$$



$$V_c(s) = \frac{1}{s} \left[\frac{1}{RC} \left(\frac{V_s(s)}{s} - V_c(s) \right) \right]$$

$$V_c(s) = \frac{1}{RCs} [V_s(s) - V_c(s)]$$

$$V_c(s) = \frac{1}{RCs} [V_s(s) - V_c(s)]$$

$$RCs V_c(s) = V_s(s) - V_c(s)$$

$$(RCs + 1) V_c(s) = V_s(s)$$

$$\rightarrow G(s) = \frac{V_c(s)}{V_s(s)} = \frac{1}{RCs + 1}$$

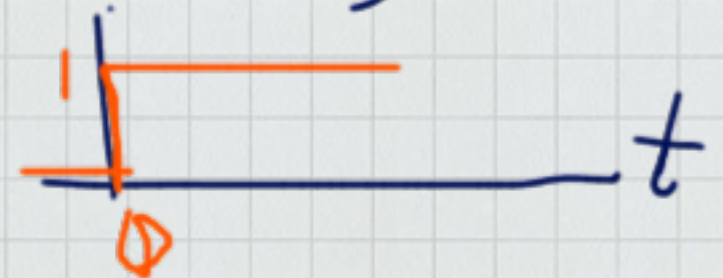
* $v_s(t) = 10u(t)$, $u(t)$: start undak satuan (unit step function)

$$* V_s(s) = \mathcal{L}\{v_s(t)\}$$

$$= \mathcal{L}\{10u(t)\}$$

$$= 10 \mathcal{L}\{u(t)\} = 10 \left[\frac{1}{s} \right] = \frac{10}{s}$$

$$u(t) \begin{cases} = 0, t < 0 \\ = 1, t \geq 0 \end{cases}$$



$$* V_c(s) = \mathcal{L}\{v_c(t)\}$$

$$= G(s) \cdot V_s(s) \rightarrow$$

$$V_c(s) = \tau(s) \cdot V_s(s) = \left(\frac{1}{RCs + 1} \right) \left(\frac{10}{s} \right) = \frac{10}{s(RCs + 1)}$$

Misalnya: $R = 100 \text{ k}\Omega$
 $C = 100 \mu\text{F}$ } $RC = 100 \times 10^3 \times 100 \times 10^{-6}$
 $\tau = 10 \text{ sec}$

$$* V_c(s) = \frac{10}{s(10s + 1)} = \frac{10}{s(\tau s + 1)} \quad \tau = \text{"tau"} \quad \text{(time constant)}$$

$$* v_c(t) = \mathcal{L}^{-1} V_c(s) = \mathcal{L}^{-1} \frac{10}{s(10s + 1)}$$

$$= \mathcal{L}^{-1} \frac{\cancel{10}}{\cancel{10}s(s + 0.1)} = \mathcal{L}^{-1} \frac{1}{s(s + 0.1)}$$

$$\frac{1}{s(s + a)} = \frac{1}{a} (1 - e^{-at}) \quad = \frac{1}{0.1} (1 - e^{-0.1t})$$

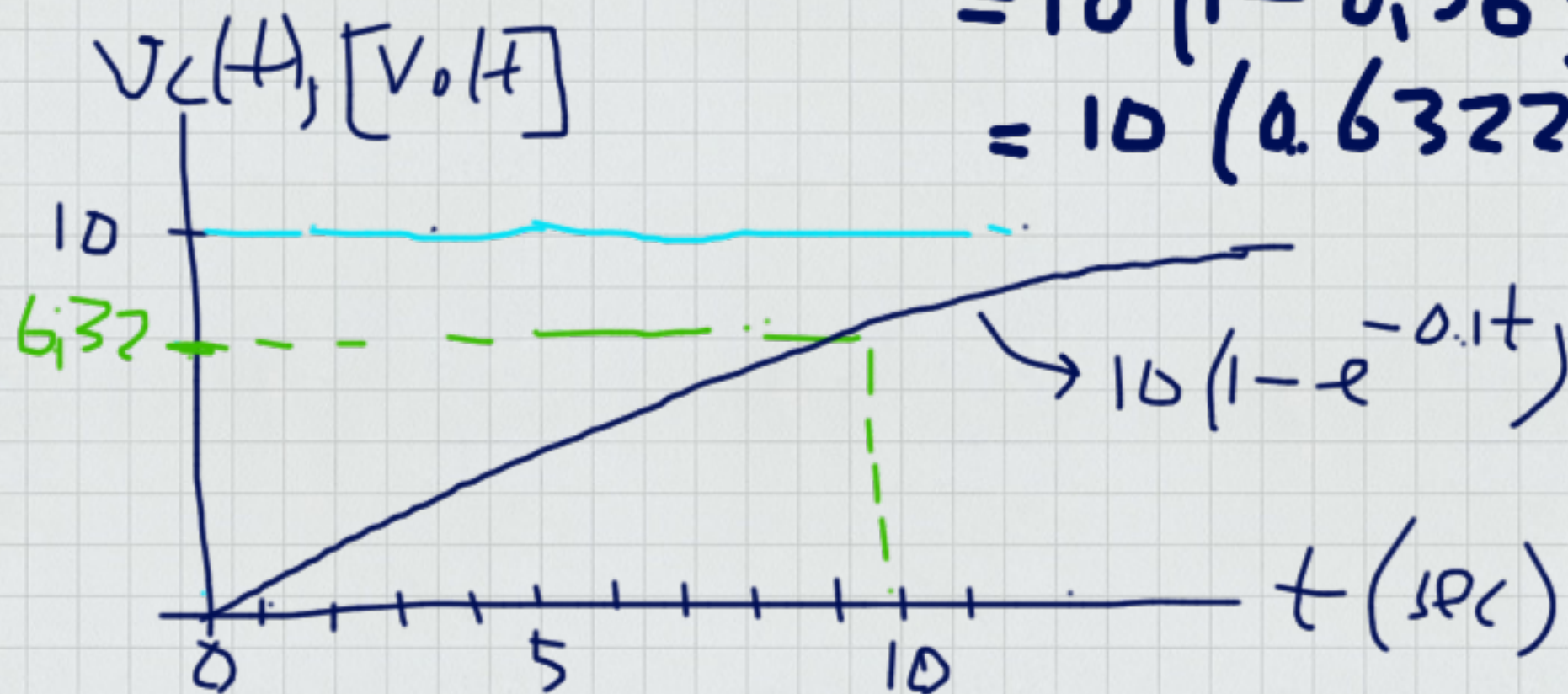
$$v_c(t) = 10(1 - e^{-0.1t})$$

check = Keadaaan awal $t=0$, $v_c(0) = 10(1 - e^{-0}) = 0$
 Keadaaan akhir $t \rightarrow \infty$, $v_c(\infty) = 10(1 - \frac{1}{\infty}) = 10$

Pada $t = \tau = 10 \rightarrow v_c(t) = 10(1 - e^{-1})$ [Volt]

$$= 10(1 - 0.3678)$$

$$= 10(0.6322) = 6.322 \text{ Volt}$$

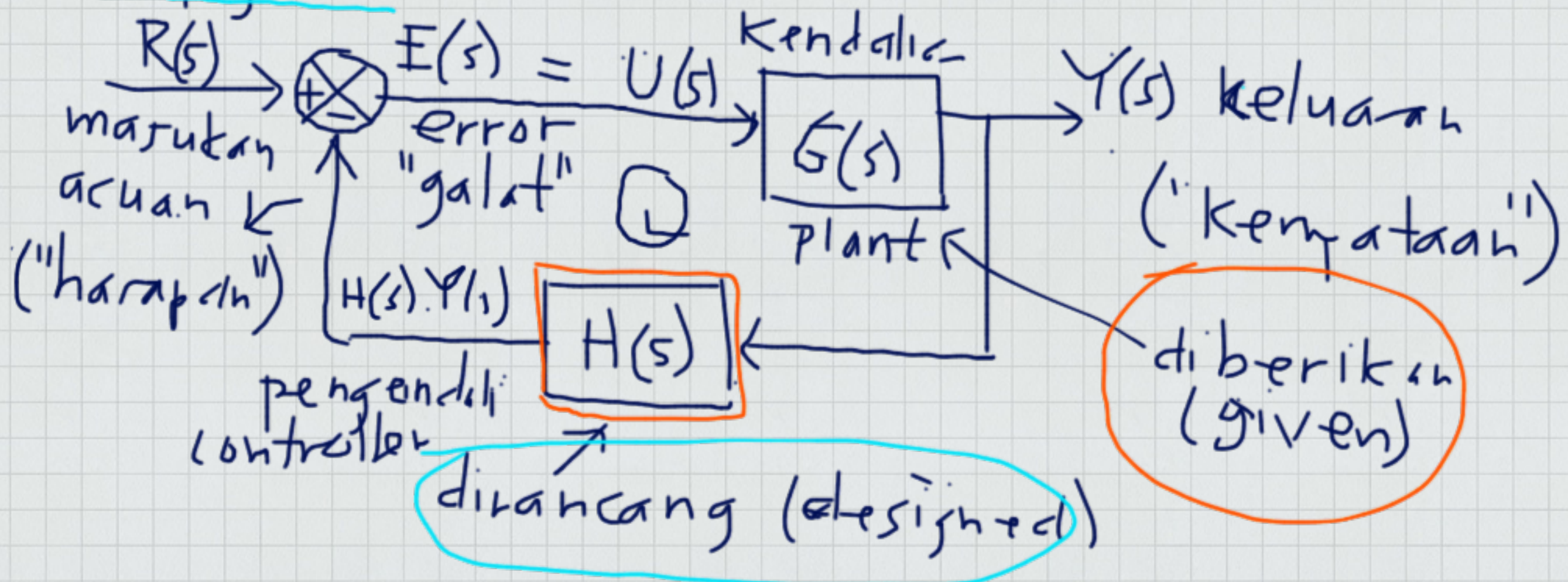


Bab I Pengenalan SISTEM KENDALI ✓
Bab II Alat² MATEMATIK ✓

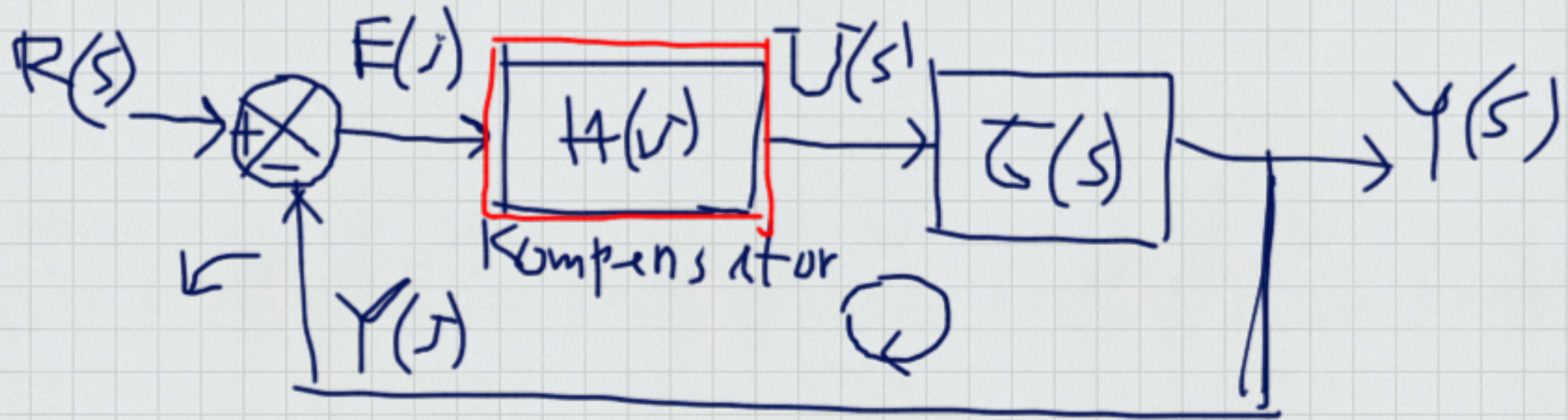
Bab III ISTILAH³ KHUSUS

* Sistem Kendali dengan Umpanbalik

Konfigurasi



* Sistem Kendali dengan Kompensator



⇒ Nisbah Alih Daur Terbuka (Open Loop
Transfer Function), sama²: $G(s) \cdot H(s)$

⇒ Nisbah Alih Daur Tertutup (Closed Loop
Transfer Function):

$$G_T(s) = \frac{Y(s)}{R(s)}$$

$\Rightarrow$ dengan umpan balik: $G_T(s) = \frac{G(s)}{1 + G(s)H(s)}$

$\Rightarrow$ dengan kompenjator: $G_T(s) = \frac{G(s)H(s)}{1 + G(s)H(s)}$

* Perjamaan Karakteristik

Penyebut
 $G_T(s)$ $\leftarrow 1 + G(s)H(s) = 0$
 polynomial dalam s

Contoh: $G(s) = \frac{1}{s^2}$ (double integrator plant)

Ditanyakan: $H(s) = \frac{s+a}{s+b}$ (phase compensator)

* OLTF

* CLTF

* Pers. Karakteristik

$\searrow$ phase lag
 phase lead

$$K_2 = \frac{K_1}{1 + 0,4K_1} = \frac{6}{1 + 2,4} = \frac{60}{34} = 1\frac{13}{17}$$

$$K = 5 + K_2 = 6\frac{13}{17}$$

Cara II Percabangan dan Pertemuan Isyarat

$$E = X - 0,4(Y - 5X) = X - 0,4Y + 2X = 3X - 0,4Y$$

$$6E = Y - 5X$$

$$6(3X - 0,4Y) = Y - 5X$$

$$18X - 2,4Y = Y - 5X$$

$$3,4Y = 23X$$

$$Y = \frac{23}{3,4}X = \frac{115}{17}X = 6\frac{13}{17}X$$

Rencana MID : 2 pekan lagi : 13/11/2015

Bahan: Bab I, II, III

Sitat: OPEN BOOK
NO LAPTOP

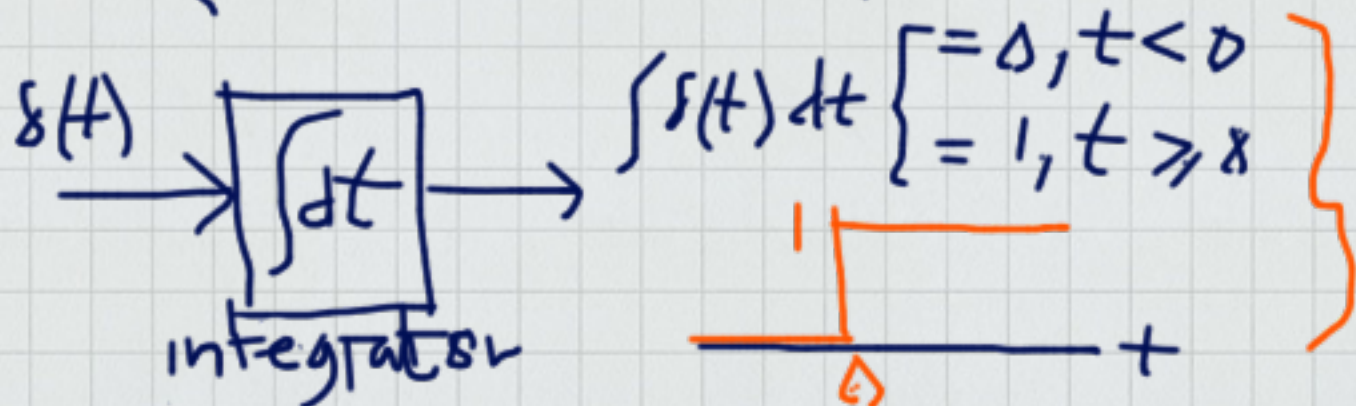
Jangan lupa membawa Tabel LAPLACE

* Isyarat uji (TEST SIGNAL)

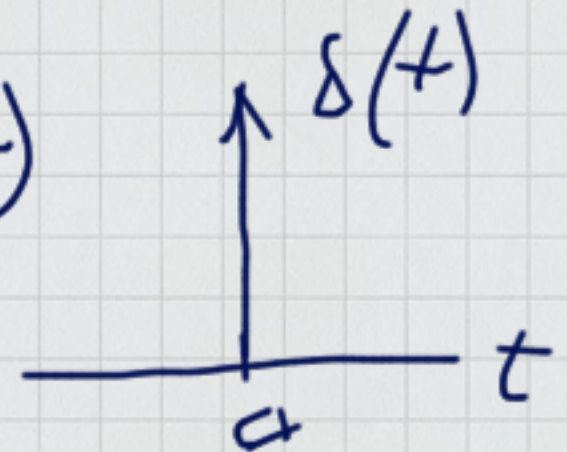
⇒ Isyarat denyut satuan, $\delta(t)$
(unit impulse)

$$L\{\delta(t)\} = 1$$

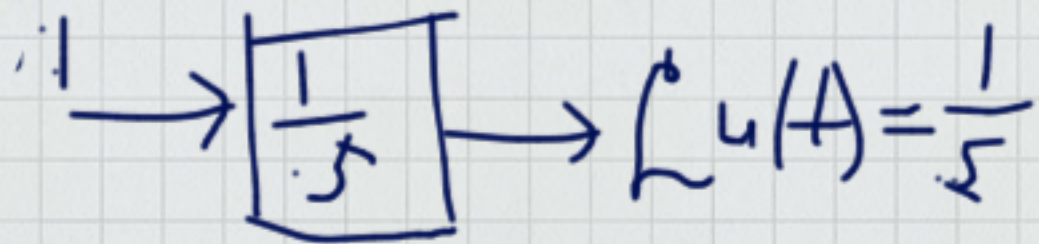
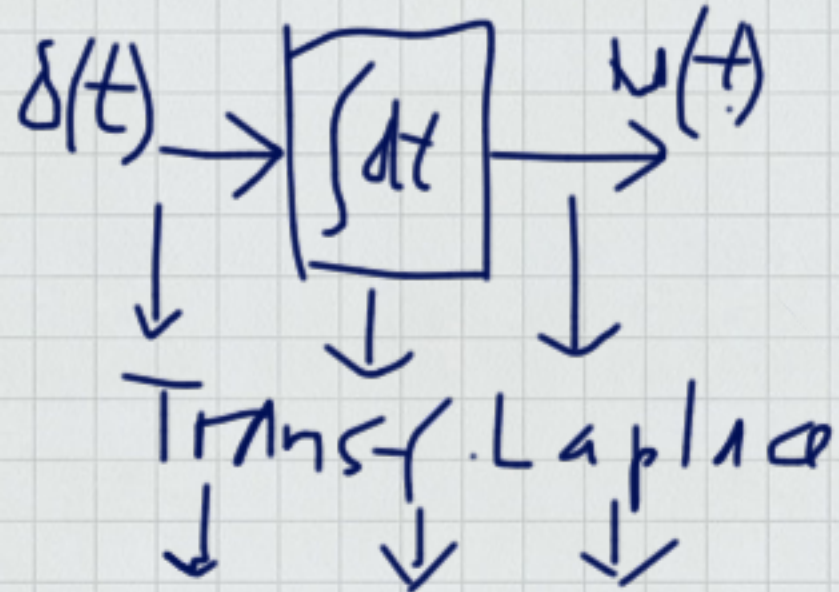
(sudah dibahas di Bab II)



Isyarat
undak satuan



* Isyarat Undak Satuan, $u(t)$ (unit step)

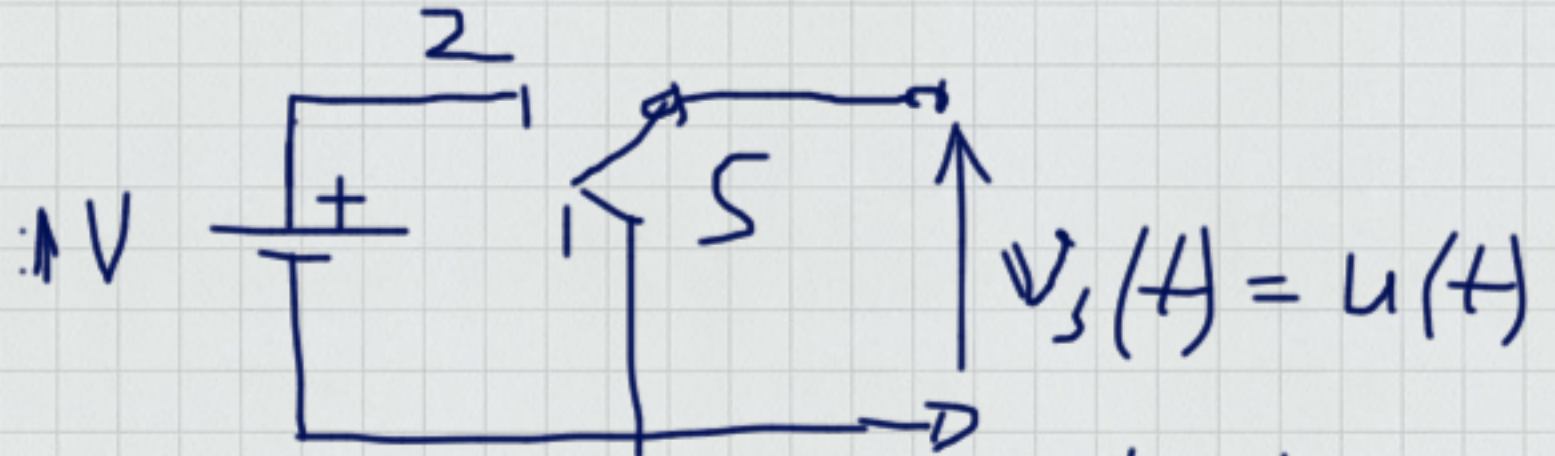


"delay" (waktu tunda)

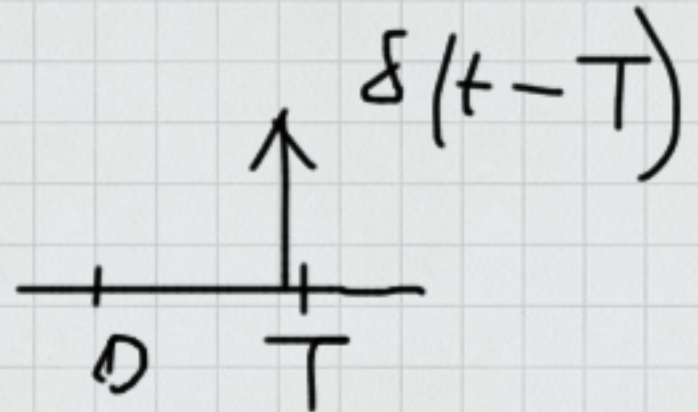


$$u(t-T) \begin{cases} = 0, & t < T \\ = 1, & t \geq T \end{cases}$$

Realisasi fisik:

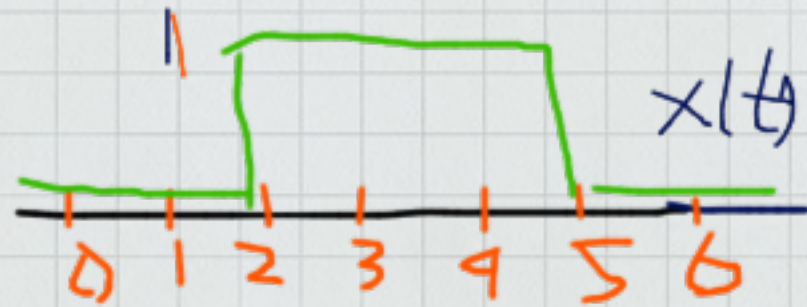


Saklar S dipindahkan dari posisi 1 ke posisi 2 seketika pada $t = 0$.



$$\mathcal{L} \delta(t-T) = -e^{-Ts}$$

$$\mathcal{L} u(t-T) = \frac{1}{s} e^{-Ts}$$



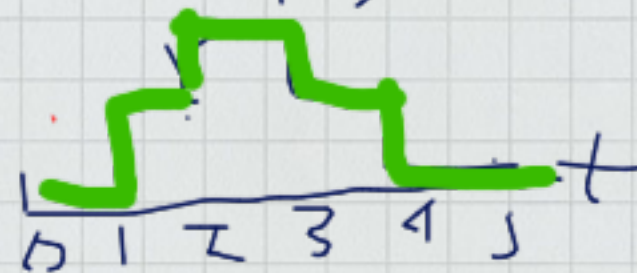
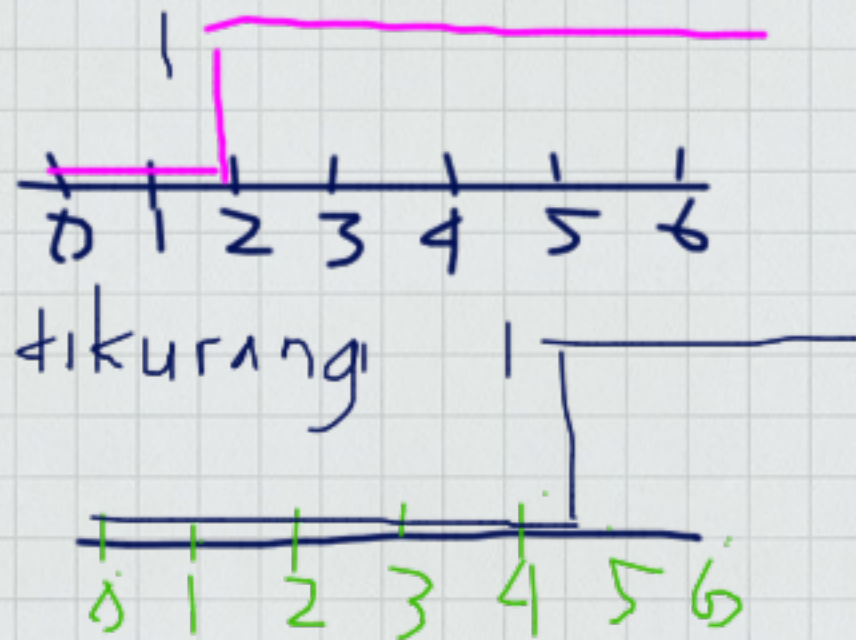
$$x(t) = u(t-2) - u(t-5)$$

$$X(s) = \mathcal{L} x(t) = \mathcal{L} [u(t-2) - u(t-5)]$$

$$= \frac{1}{s} (e^{-2s} - e^{-5s})$$

Latihan

* Tentukan $x(t)$ dan $X(s)$.



* Gambar $y(t) =$
 $5u(t-1) - 3u(t-2) - 2u(t-3)$

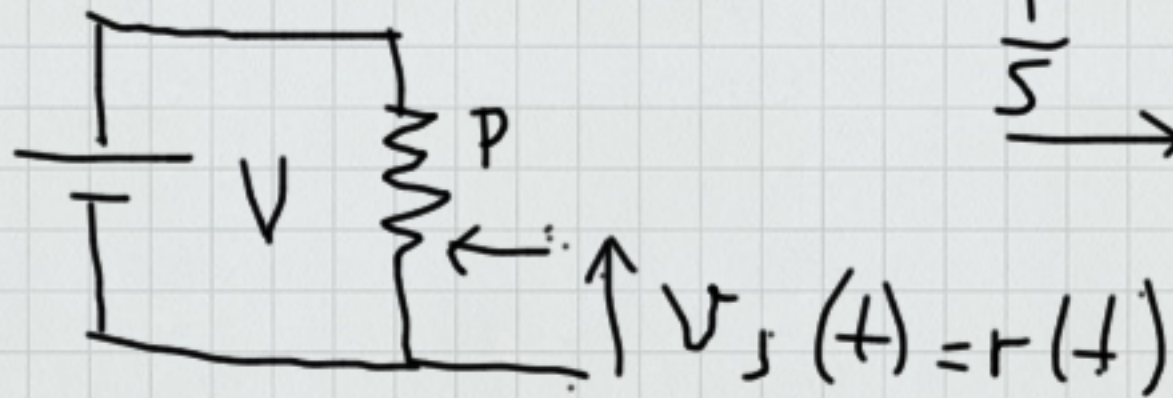
* Isyarat tanjakan satuan $r(t)$
(unit ramp)

$$r(t) = \int u(t) dt \begin{cases} = 0, t < 0 \\ = t, t \geq 0 \end{cases}$$

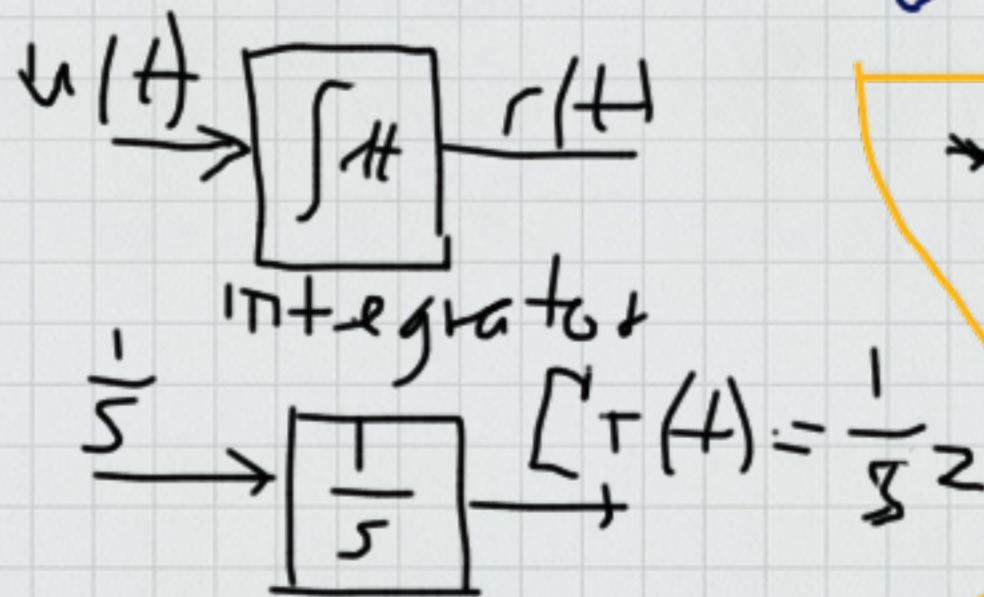


Note = $\int dt = t$

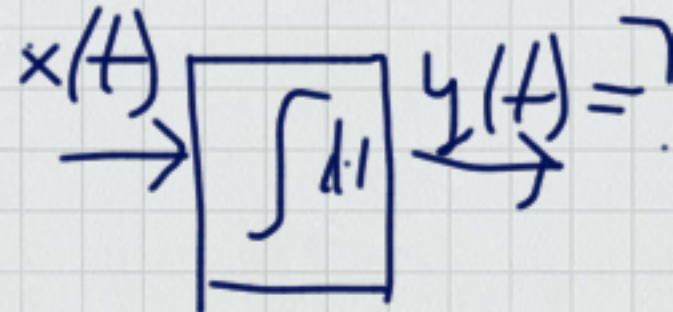
Realisasi fisik



Potensiometer P digeser naik
sbg menghasilkan $V_s(t) = r(t)$
Yang naik 1V setiap 1 detik



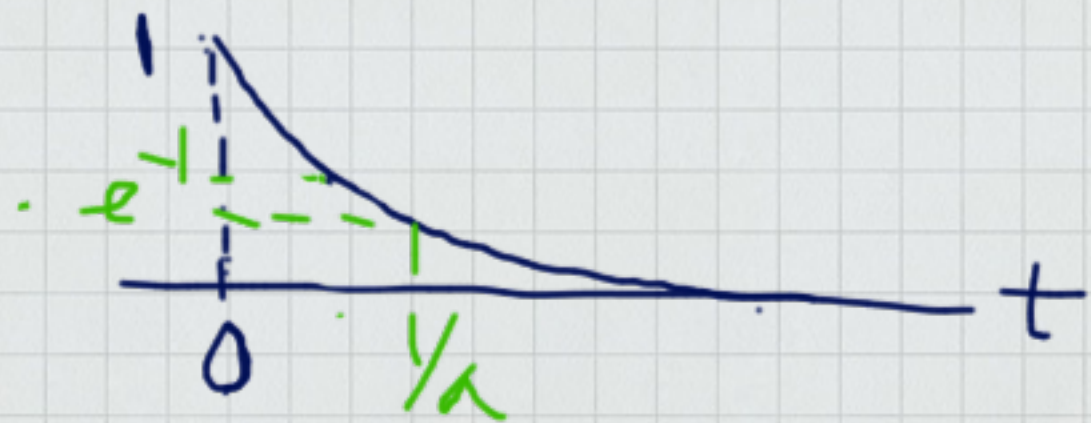
* Untuk Latihan.



$$x(t) = u(t) - u(t-T)$$



* Isyarat exponential e^{-at} , $t \geq 0$

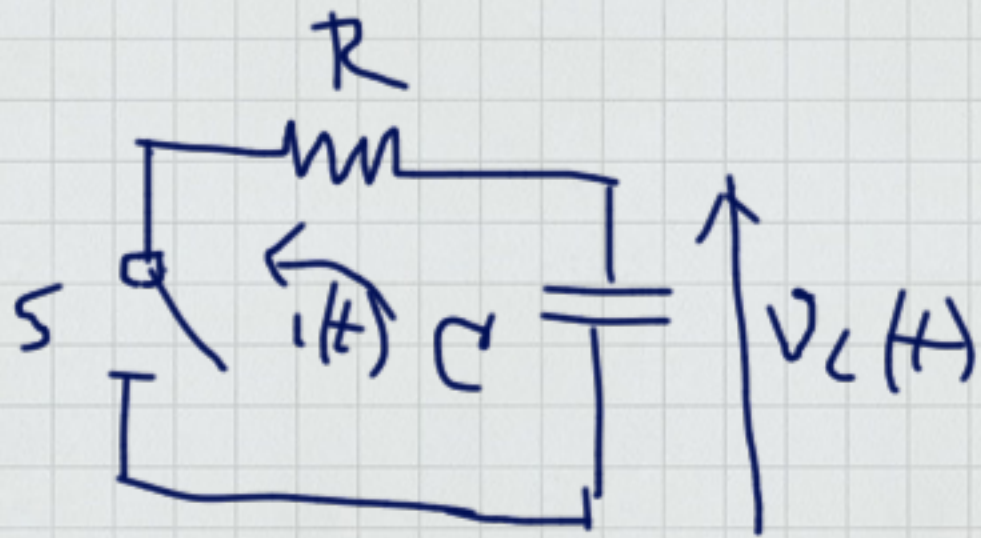


$$t = 0 \rightarrow e^{-at} = 1$$

$$t = \frac{1}{a} \rightarrow e^{-1} = 0,3678$$

$= \tau$ (tau), time constant

$$t \rightarrow \infty \rightarrow e^{-\infty} = 0$$



Pada $t = 0$, $V_C(0) = V$ jika S ditutup

$$V_C(t) = -Ri(t) = -R \frac{dV_C(t)}{dt}$$

$$\int \frac{dV_C(t)}{V_C(t)} = -\frac{t}{RC} + K \rightarrow \ln V_C(t) = -\frac{t}{RC} + K$$

$$V_c(t) = e^{\left(-\frac{t}{RC} + K\right)} = A e^{-t/RC}$$

$$t=0 \rightarrow V_c(0) = A = V$$

$$V_c(t) = V e^{-t/RC}$$

$$\left. \begin{array}{l} V = 1 \text{ Volt} \\ a = 1/RC \end{array} \right\} V_c(t) = e^{-at}$$

$$\int_0^\infty e^{-at} = \frac{1}{s+a}$$

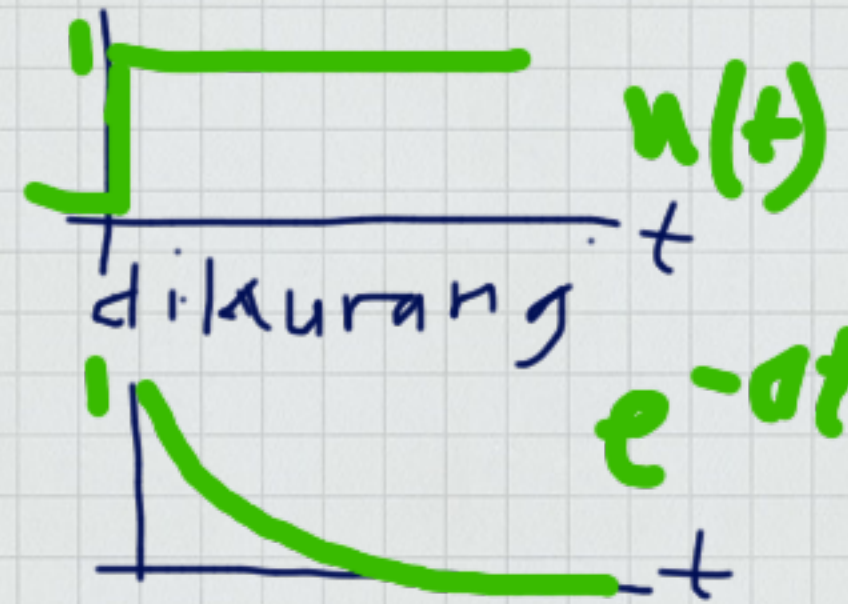
* Latihan: $x(t) = 1 - e^{-at}, t \geq 0$

$\Rightarrow$ Bagaimana bentuk $x(t)$?

$\Rightarrow$ tentukan $X(s) = \mathcal{L}\{x(t)\}$!

$$\mathcal{L}\{x(t)\} = \mathcal{L}\{1 - e^{-at}\} = \frac{1}{s} - \frac{1}{s+a} = \frac{a}{s(s+a)}$$

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s+a)}\right\} = \frac{1}{a}(1 - e^{-at})$$



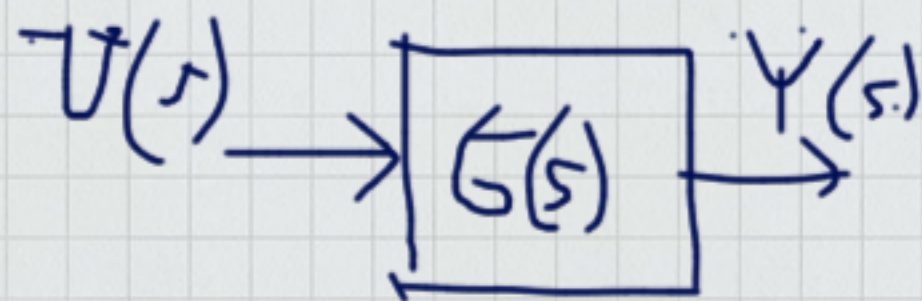
MIDTEST : PEKANI DEPANI: 12/11/15

* Bahan: Bab I, II, III

* OPEN BOOK, NO LAPTOP

* Jangan lupa membawa TABEL LAPLACE

* POLE, ZERO dan ORDER



$$G(s) = \frac{Y(s)}{U(s)}$$

$$G(s) \equiv \frac{\text{Polynomial } s \text{ berderajat } m, N(s)}{\text{Polynomial } s \text{ berderajat } n, D(s)}$$

$$m \leq n$$

[Kasus $m > n$ adalah HPF (High Pass Filter) yang "physically unrealizable"]

$$N(s) = b_m s^m + b_{m-1} s^{m-1} + b_{m-2} s^{m-2} + \dots + b_1 s + b_0$$

$$D(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0$$

$m \leq n$, koefisien $a_0, \dots, a_n, b_0, \dots, b_m \in \text{Real}$

* Akar² pers. $N(s) = 0$ disebut ZERO, z_1 s/d z_m

* Akar² pers. $D(s) = 0$ disebut POLE, p_1 s/d p_n

POLE dan ZERO dapat dipetakan pada

BIDANG KOMPLEKS masing² dengan

notasi $\times$ untuk pole dan $\bigcirc$ untuk zero

* ORDER dari $G(s)$ adalah order ke-n.

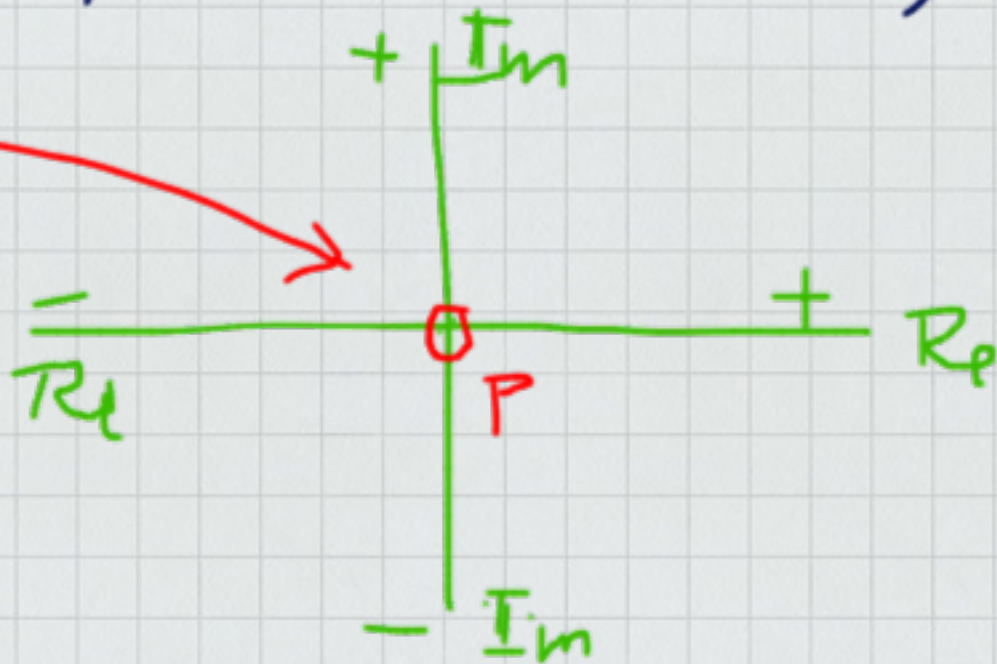
$\Rightarrow$ * Order dari $G(s)$ sesuai dengan jumlah pole-nya

Terkait dengan konfigurasi dasar:

- ⇒ Order dari sistem kendali sesuai dengan jumlah pole $G(s)$ + jumlah pole $H(s)$.
- ⇒ Pole dan zero dari OLTF $G(s) \cdot H(s)$ adalah pole dan zero dari masing-masing $G(s)$ dan $H(s)$.
- ⇒ Pole dari CLTF $G_T(s)$ adalah akar-akar pers. karakteristik
$$1 + G(s)H(s) = 0$$

Contoh: Suatu sistem kendali mempunyai
Kendalian $G(s) = \frac{1}{s}$ dan $H(s) = \frac{s+1}{s+2}$
(phase compensator) sebagai pengent-
datannya

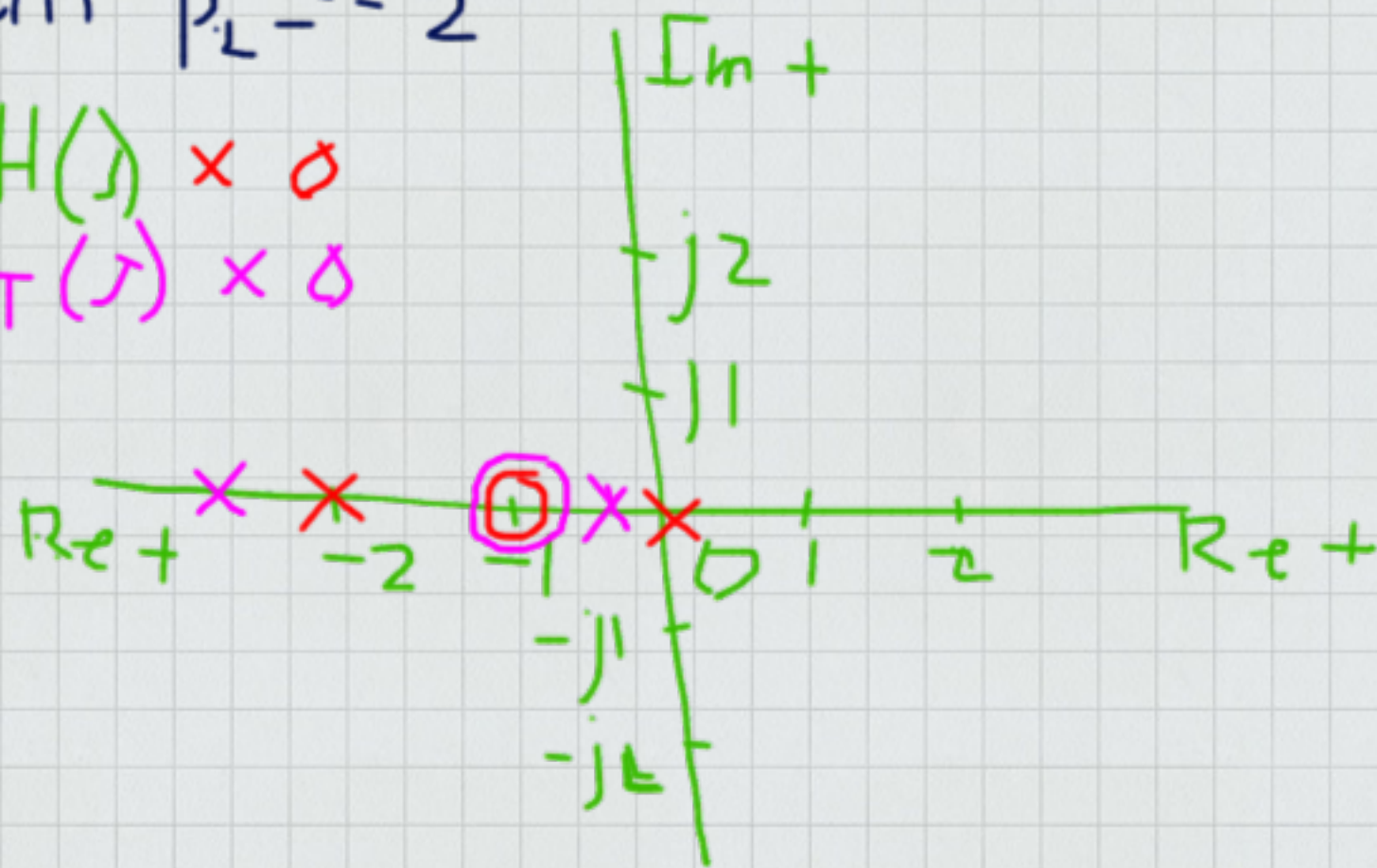
* $G(s)$ sistem order pertama (1st order)
yg tidak punya zero, dengan pole
 $p=0$



* $H(s)$ adalah sistem order pertama dengan ZERO $z = -1$, dan POLE $p = -2$

* OLTF $G(s) \cdot H(s)$ adalah sistem order kedua, dengan $z = -1$, dan 2 POLE-nya $p_1 = 0$ dan $p_2 = -2$

$$G(s)H(s) \times 0$$
$$G_T(s) \times 0$$



* CLTF $G_T(s) = \frac{G(s) \cdot H(s)}{1 + G(s) H(s)}$ system kendali
derigan
KOMPENSATOR

$$= \frac{\frac{1}{s} \cdot \frac{s+1}{s+2}}{1 + \frac{1}{s} \cdot \frac{s+1}{s+2}} = \frac{s+1}{s^2 + 2s + s + 1} = \frac{s+1}{s^2 + 3s + 1}$$

memiliki 1 zero $z = -1$

2 pole : $p_{1,2} = \frac{-3 \pm \sqrt{9-4}}{2}$

$$p_1 = -1,5 + 1,18 = -0,382$$

$$p_2 = -1,5 - 1,18 = -2,618$$

$$= -\frac{3}{2} \pm \frac{1}{2}\sqrt{5}$$

* Pengendali PID

P = Proportional
I = Integral
D = Differential

$$H(s) = K_p + \frac{K_I}{s} + K_D s$$

* $K_p = 0 \rightarrow PI \rightarrow H(s) = \frac{K_I s + K_I}{s}$ order pertama
 $p = 0$
 $z = -\frac{K_I}{K_p}$

* $K_I = 0 \rightarrow PD \rightarrow H(s) = K_p + K_D s$ $\rightarrow m > n$

(tidak dapat direalisasi sendiri)

* $K_D = K_I = 0 \rightarrow P \rightarrow H(s) = K_p$ order nol

* $\{K_p, K_I, K_D\} \neq 0 \rightarrow PID \rightarrow H(s) = \frac{K_p s + K_I + K_D s^2}{s}$

Order Pertama: $m > n \rightarrow 2 \text{ zero}$
 5
 1 pole

$G(s)$ minimal: $G(s) = \frac{1}{s}$

* Sistem² order pertama:

$\Rightarrow G(s) = \frac{1}{s}$, integrator

$\Rightarrow G(s) = \frac{1}{s+a}$, lagging, (simple lag)

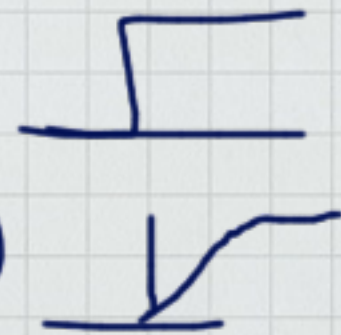
$$G(s) = \frac{1}{\tau s + 1}$$

$\Rightarrow G(s) = \frac{s+a}{s+b}$, phase compensator

$\tau = "tau"$

$a < b \Rightarrow$ phase-lag

$a > b \Rightarrow$ phase-lead



$$\tau = \frac{1}{a}$$

time
constant

* Sistem² order kedua:

$\Rightarrow G(s) = \frac{1}{s^2}$, double integrator

$\Rightarrow G(s) = \frac{1}{s(s+a)}$, simple (lag + integrator)

* Sistem² order pertama:

$\Rightarrow G(s) = \frac{1}{s}$, integrator

$\Rightarrow G(s) = \frac{1}{s+a}$, lagging, (simple lag)

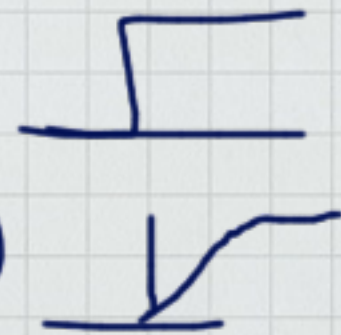
$$G(s) = \frac{1}{\tau s + 1}$$

$\Rightarrow G(s) = \frac{s+a}{s+b}$, phase compensator

$\tau = "tau"$

$a < b \Rightarrow$ phase-lag

$a > b \Rightarrow$ phase-lead



$$\tau = \frac{1}{a}$$

time constant

* Sistem² order kedua:

$\Rightarrow G(s) = \frac{1}{s^2}$, double integrator

$\Rightarrow G(s) = \frac{1}{s(s+a)}$, simple (lag + integrator)

Bentuk Umum Sistem Order Kedua.

$$G(s) = \frac{K}{s^2 + 2\zeta\omega_n s + \omega_n^2}$$

ζ : zeta : nisbah redaman (damping ratio)

ω_n : omega.n : frekuensi (sudut, radian/sec) alamiah tak teredam (undamped natural frequency)

Sistem Order Kedua karakteristiknya tergantung pada nilai ζ

* $\zeta > 1 \rightarrow$ teredam lebih (overdamped)
 2 pole Real: $G(s) = \frac{K}{(s+a)(s+b)}$

* $\zeta = 1 \rightarrow$ teredam kritis (critically damped)
 2 pole Real KEMBAR

* $0 < \zeta < 1 \rightarrow$ teredam kurang (under damped)
 2 pole Pasangan Komplek

Konjugasi
 $p_{1,2} = a \pm jb$
 $j = \sqrt{-1}$

* $\zeta = 0 \rightarrow$ Undamped oscillation

* $\zeta < 0$
 2 pole pasangan imajiner
 $p_{1,2} = \pm jb$
 $\hookrightarrow$ UNSTABLE

* $\zeta > 1 \rightarrow$ teredam lebih (overdamped)
 2 pole Real: $G(s) = \frac{K}{(s+a)(s+b)}$

* $\zeta = 1 \rightarrow$ teredam kritis (critically damped)
 2 pole Real KEMBAR

* $0 < \zeta < 1 \rightarrow$ teredam kurang (under damped)
 2 pole Pasangan Komplek

Konjugasi
 $p_{1,2} = a \pm jb$
 $j = \sqrt{-1}$

* $\zeta = 0 \rightarrow$ Undamped oscillation

* $\zeta < 0$
 2 pole pasangan imajiner
 $p_{1,2} = \pm jb$
 $\hookrightarrow$ UNSTABLE

Tentukan tanggapan denyut dari

$$* G(s) = \frac{10}{s^2 + 4s + 4} \rightarrow \xi = 1$$

$$* G(s) = \frac{10}{s^2 + 8s + 4} \rightarrow \xi > 1$$

$$* G(s) = \frac{10}{s^2 + s + 4} \rightarrow 0 < \xi < 1$$

$$* G(s) = \frac{10}{s^2 + 4} \rightarrow \xi = 0$$

Note: Gunakan Tabel Laplace

Soal MID :

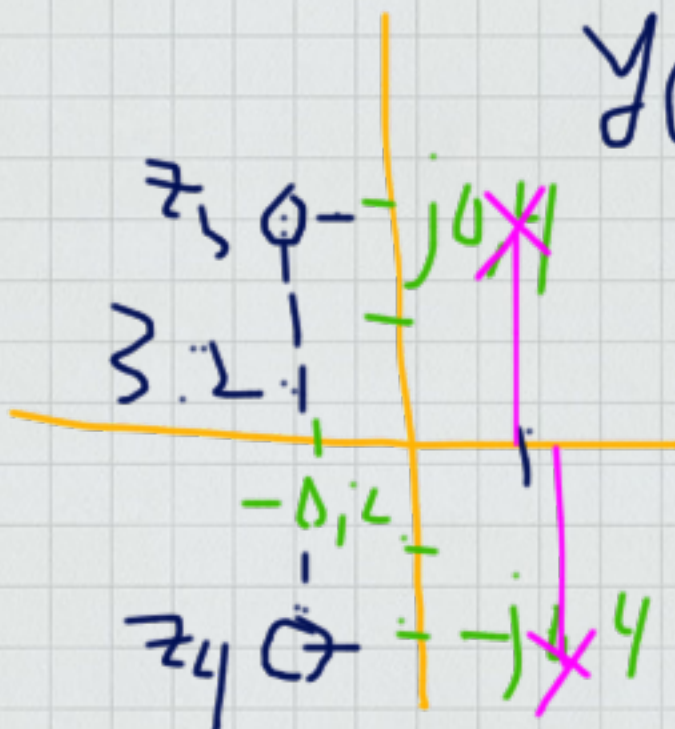
2.2. $\frac{d^2 y(t)}{dt^2} + 6 \frac{dy(t)}{dt} + 9 y(t) = x(t)$

2.3. $G(s) = \frac{Y(s)}{X(s)} = \frac{1}{s^2 + 6s + 9} = \frac{1}{(s+3)^2}$

2.4. $x(t) = \delta(t) \rightarrow X(s) = 1$

$$Y(s) = G(s)X(s) = \frac{1}{(s+3)^2}$$

$$y(t) = \mathcal{L}^{-1} Y(s) = \mathcal{L}^{-1} \frac{1}{(s+3)^2} = te^{-3t}$$



$$z_{3,4} = \frac{-2 \pm \sqrt{4 - 20}}{10} = -0,2 \pm j0,4$$

$$p_{4,5} = \frac{2 \pm \sqrt{4 - 20}}{10} = 0,2 \pm j0,4$$

* ANALISIS KESTABILAN

Tujuan utama dari pengendalian sistem adalah untuk men-STABIL-kan sistem tersebut.

Setelah sistem stabil, barulah dilakukan hal-hal lain, misalnya untuk:

- * Meng-OPTIMUM-kan kinerja =
 - Me-minimum-kan biaya (cost)
 - Me-maximum-kan keuntungan (benefit)
 - Me-maximum-kan kinerja (performance)
- SISTEM KENDALI OPTIMAL

* Membuat sistem tidak terpengaruh pada perubahan lingkungan, atau mampu menyesuaikan dengan lingkungan

⇒ SISTEM KENDALI ADAPTIF

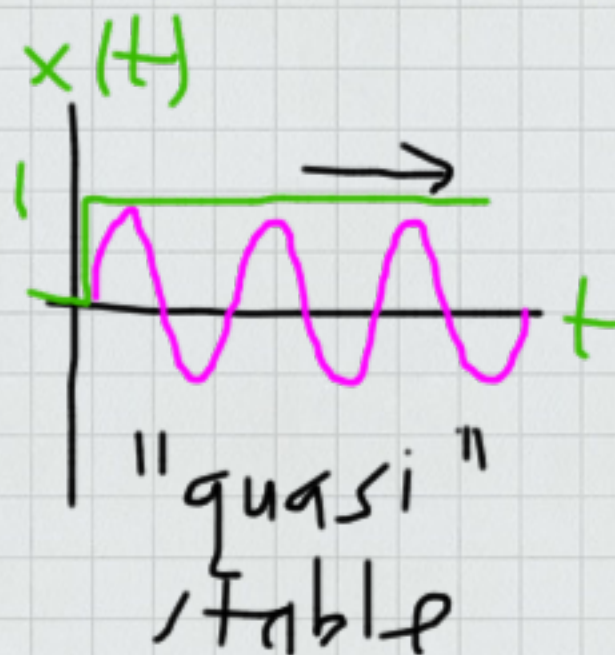
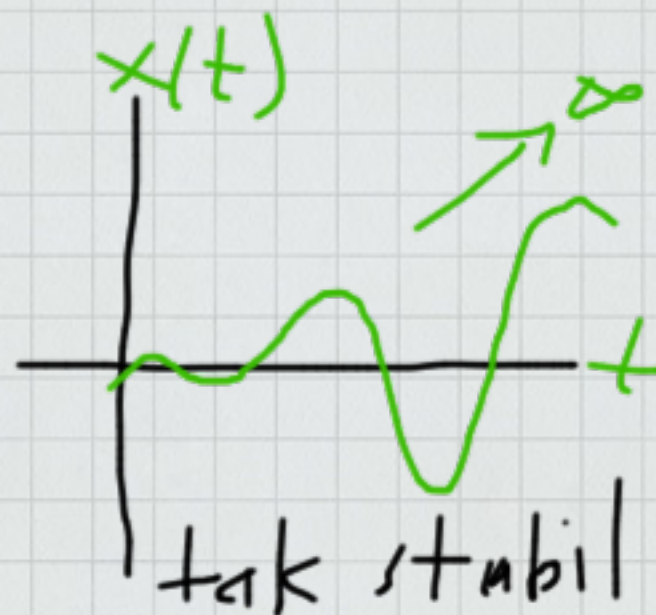
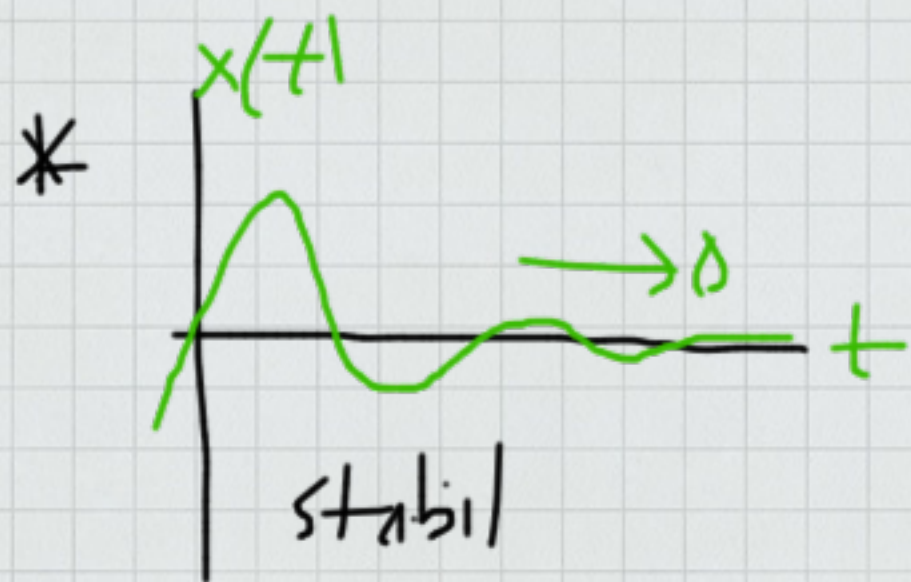
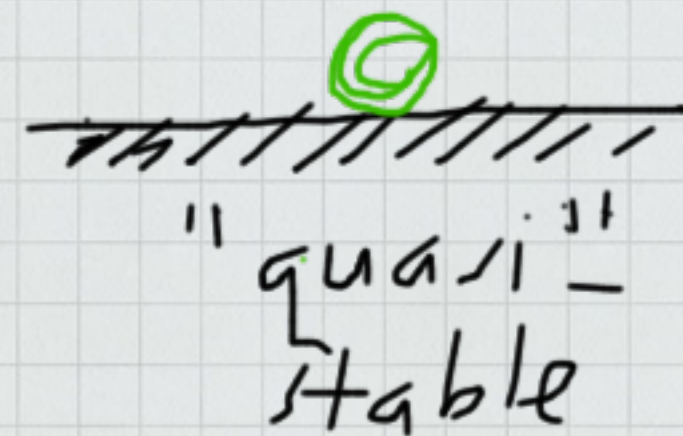
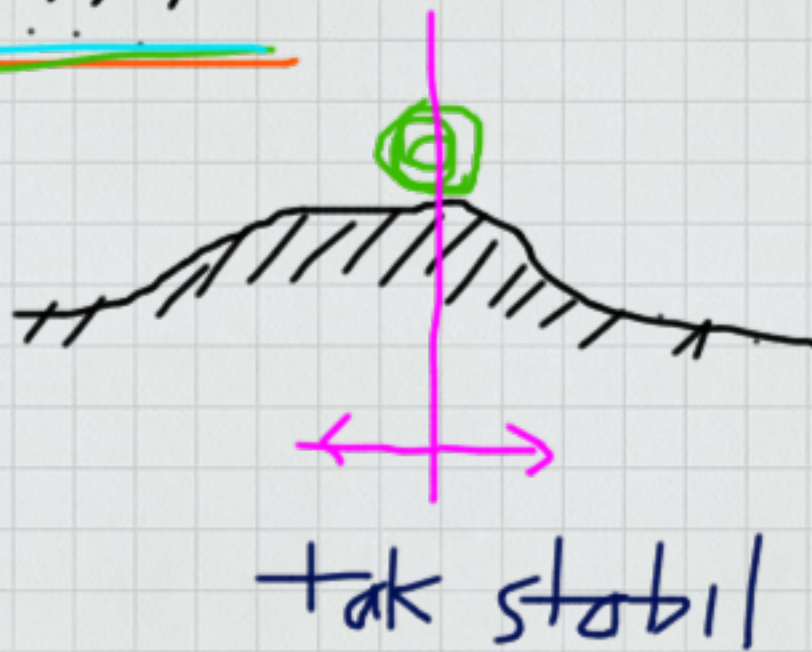
* Mampu mengatasi perubahan "internal" (penurunan-aging, keausan, perubahan setting, kalibrasi, dll)

⇒ SISTEM KENDALI KOKOH (robust)

* Mampu "belajar", bisa "dilatih"

SISTEM KENDALI "CERDAS"

* STABIL - - - ???



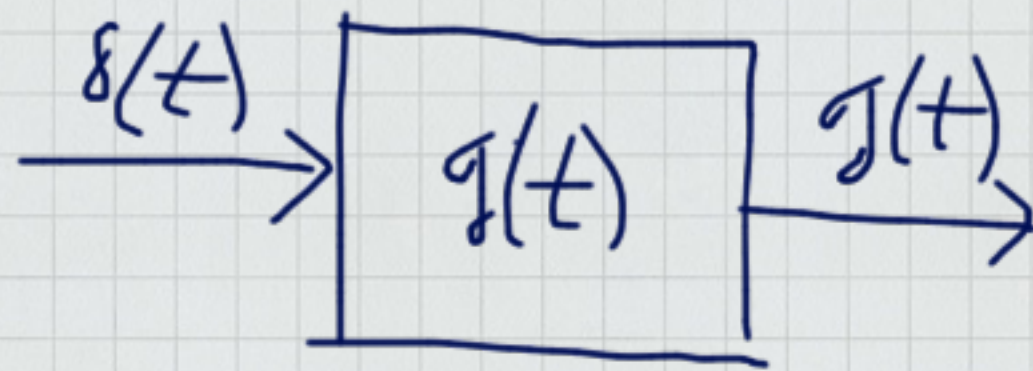
Ada banyak definisi kestabilan, misalnya:

- * Asymptotic stability
- * BIBO stability

BIBO = Bounded Input Bounded Output

Dalam m.k. ini, kestabilan didefinisikan dengan tanggapan deniyut (impulse response)

[Suatu sistem kendali atau kendalian dikatakan STABIL jika (dan hanya jika) tanggapan deniyutnya menuju nol]

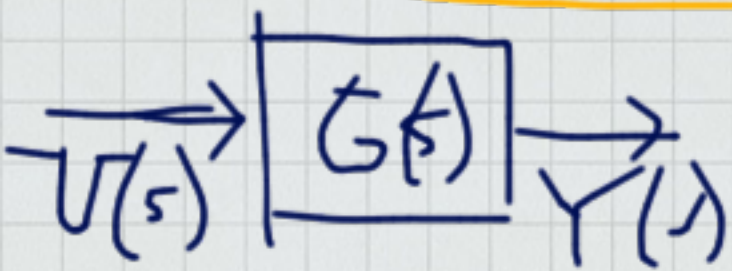


sistem $g(t)$
stabil, jika

$$\lim_{t \rightarrow \infty} g(t) = 0$$

Kestabilan sistem dapat dilihat dari Nisbah Alih sistem tsb.

* Sistem Order Pertama



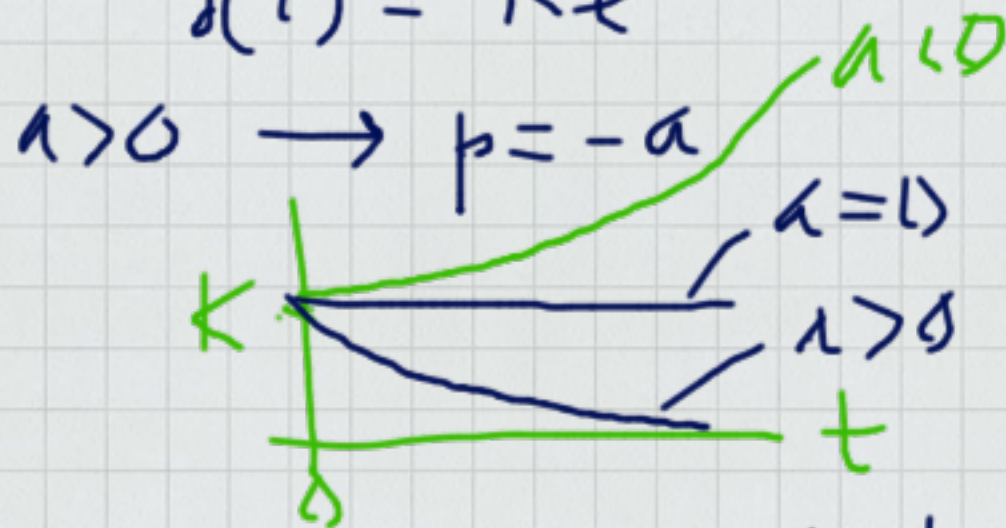
$$G(s) = \frac{K}{s+a}$$

$$u(t) = \delta(t)$$

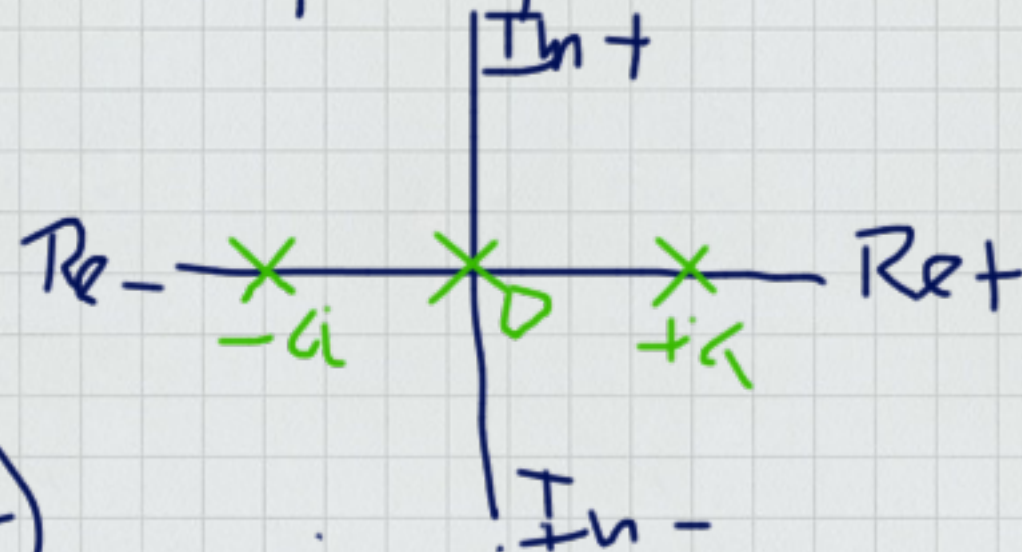
$$U(s) = 1$$

$$Y(s) = G(s) = \frac{K}{s+a}$$

Tanggapan denrit $y(t) = \mathcal{L}^{-1} Y(s) = \mathcal{L}^{-1} \frac{K}{s+a}$
 $y(t) = K e^{-at}$
 - "Simple-pole"



$a = 0 \rightarrow p = 0$ (integrator)
 $a < 0 \rightarrow p = +a$



KE SIMPULAN $\Rightarrow$

TARGET FINAL
 10/12/15
 Materi MID +
 Bab IV

Suatu simple-pole (Sistem Order Pertama) hanya akan stabil jika pole-nya berada di sebelah kiri sumbu khayal pada bidang kompleks

Untuk latihan = Bgm dengan "Phase Compensator" : $G(s) = K \frac{s+a}{s+b}$???

Hint : $G(s) = K + G'(s)$

$G'(s) = \frac{K(a-b)}{s+b} = \frac{K'}{s+b}$

$G'(s) = \frac{K}{s+b} - \frac{Ks+K_a}{s+b} = \frac{K(a-b)}{s+b}$

→ Simpulkan sendiri - - -

* Sistem Order Kedua

$$G(s) = \frac{K}{s^2 + 2\xi\omega_n s + \omega_n^2}$$

* $\xi > 1 \rightarrow$ $G(s) = \frac{K}{(s+a)(s+b)}$ double-pole

$$(s+a)(s+b) = 0 \quad a, b = \frac{2\xi\omega_n \pm \sqrt{4\xi^2\omega_n^2 - 4\omega_n^2}}{2}$$

$$\begin{aligned} p_1 &= -a \\ p_2 &= -b \end{aligned}$$

$$s_1 = -a$$

$$s_2 = -b$$

$$s_{1,2} = \frac{-B \pm \sqrt{B^2 - 4AC}}{2A}$$

$$= \xi\omega_n \pm \omega_n \sqrt{\xi^2 - 1} \in \text{Real}$$

$$G(s) = \frac{K_1}{s+a} + \frac{K_2}{s+b}$$

(theorem Heaviside)

$$\frac{K}{(s+a)(s+b)} = \frac{(K_1+K_2)s + K_1b + K_2a}{(s+a)(s+b)}$$

$$K_1 + K_2 = 0 \rightarrow K_1 = -K_2$$

$$K_1b + K_2a = K_1b - K_1a = K$$

$$\begin{aligned} K_1 &= \frac{K}{b-a} \\ K_2 &= \frac{K}{a-b} \end{aligned}$$

$$(s+1)(s+2)=0 \rightarrow a=1 \quad b=2$$

$$s^2 + 3s + 2 = 0$$

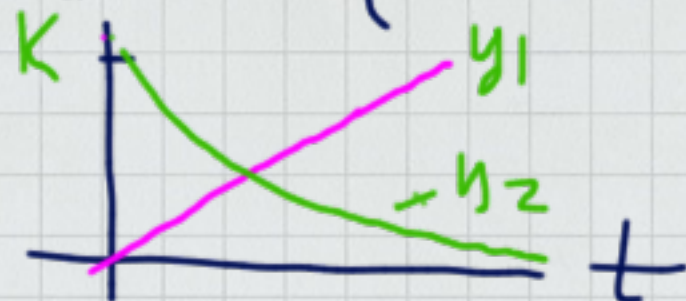
$$a, b = \frac{-3 \pm \sqrt{9-8}}{2} = \frac{-3 \pm 1}{2} \rightarrow \begin{cases} a=2 \\ b=1 \end{cases}$$

Jadi, supaya $G(s)$ stabil, kedua pole-nya harus berada di sebelah kiri sumbu khayal pada bidang kompleks.

* $\sum q = 1 \rightarrow$ pole kembar $G(s) = \frac{K}{(s+a)^2}$
 Tanggapan denyutnya: $y(t) = Kt e^{-at}$

$a > 0 \rightarrow p_1 = p_2 = -a$ (di sebelah kiri sumbu khayal)

Sistem stabil

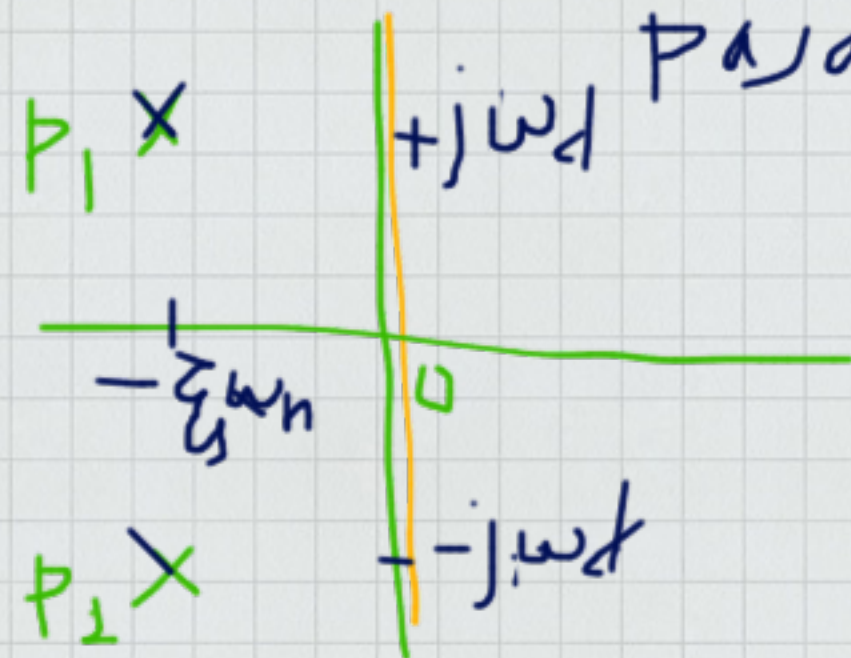


$$y(t) = y_1 \cdot y_2$$

$$t \rightarrow \infty \rightarrow \begin{cases} y_1 \rightarrow \infty \\ y_2 \rightarrow 0 \end{cases}$$

Kedua pole kembar di sebelah kiri sumbu khayal

$$* 0 < \xi < 1 \rightarrow p_{1,2} = -\xi \omega_n \pm j \omega_n \sqrt{1 - \xi^2} - \omega_d$$



pasangan kompleks konjugat

Tanggapan deniyut

$$y(t) = \frac{K}{\omega_d} e^{-\xi \omega_n t} \sin \omega_d t$$

$$\omega_d = \omega_n \sqrt{1 - \xi^2}$$

$$\xi \omega_n = 0, p_{1,2} = \pm j \omega_d \rightarrow y(t) = \frac{K}{\omega_d} \sin \omega_d t$$

$$\xi = 0$$

$$= \pm j \omega_n \rightarrow y(t) = \frac{K}{\omega_n} \sin \omega_n t$$

↳ Tidak stabil

$$\xi \omega_n < 0 \quad e^{-\xi \omega_n t} \rightarrow \infty \rightarrow \text{Tidak stabil}$$

$$\xi \omega_n > 0 \rightarrow t \rightarrow \infty \quad y(t) \rightarrow 0 \quad \text{STABIL}$$

$p_{1,2}$ di sebelah kiri sumbu khayal $\Rightarrow$ KESIMPULAN

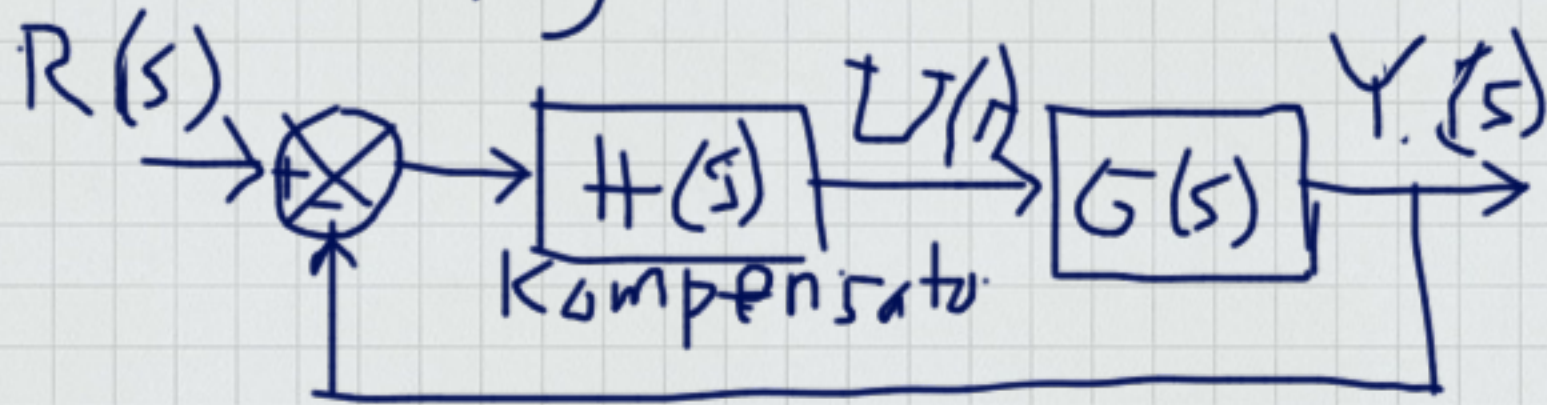
Suatu Sistem Order Kedua hanya akan stabil jika kedua pole-nya berada di sebelah kiri sumbu khayal pada bidang kompleks.

Kesimpulan umum:

Suatu sistem dengan order yang lebih tinggi, selalu dapat diuraikan menjadi perjumlahan dari sistem order pertama dan/atau sistem order kedua, maka sistem order yang lebih tinggi hanya akan stabil jika SEMUA pole-nya berada di sebelah kiri sumbu khayal pada bidang kompleks.

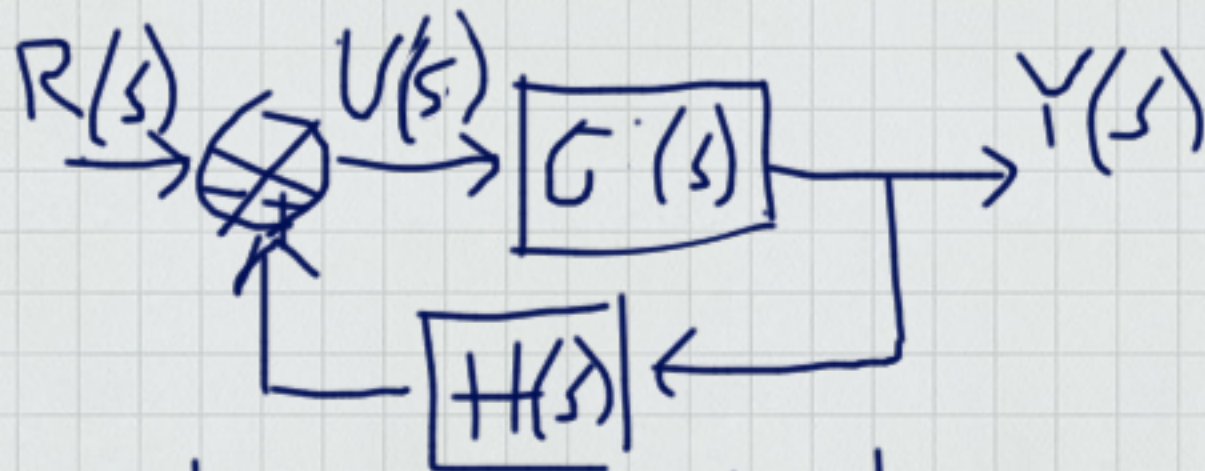
- next: Ke
Aplikasi Ke
Sistem_Kendali

Dua konfigurasi sistem kendali.



Sistem Kendali dengan
KOMPENSATOR

$$\text{CLTF} \\ G_T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)H(s)}{1 + G(s)H(s)}$$



Sistem Kendali dengan
UMPAH BALIK

$$\text{CLTF} = G_T(s) = \frac{Y(s)}{R(s)} = \frac{G(s)}{1 + G(s)H(s)}$$

Persamaan karakteristik: $1 + G(s)H(s) = 0$

Pole² dari kedua konfigurasi adalah AKAR² -
 $G_T(s), C_{LTF}$

Persamaan karakteristiknya = $1 + G(s)H(s) = 0$

Jadi =

Suatu sistem kendali akan stabil jika SEMUA akar persamaan karakteristiknya berada di sebelah kiri sumbu khayal pada bidang kompleks atau mempunyai bagian Real yang negatif ($Re < 0$).

Pada umumnya sistem kendali yang tergolong sistem LTI (Linear Time Invariant), persamaan karakteristiknya dapat dinyatakan dengan:

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0 = 0$$

dengan koefisien $a_0, a_1, \dots, a_n$ semua bilangan Real.

* Akar²nya = $s_1, s_2, s_3, \dots, s_n$, sehingga

$$(s - s_1)(s - s_2)(s - s_3) \dots (s - s_n) = 0$$

bisa dicari dengan berbagai metode, analitik maupun numerik. (Misalnya, MATLAB, `>>> root`, Mathematica, Maple dsb.)

* Salah satu cara untuk mengetahui apakah SEMUA akar ada di sebelah kiri sumbu khayal atau tidak (sistem stabil atau tidak), dapat digunakan 2 Kriteria
ROUTH

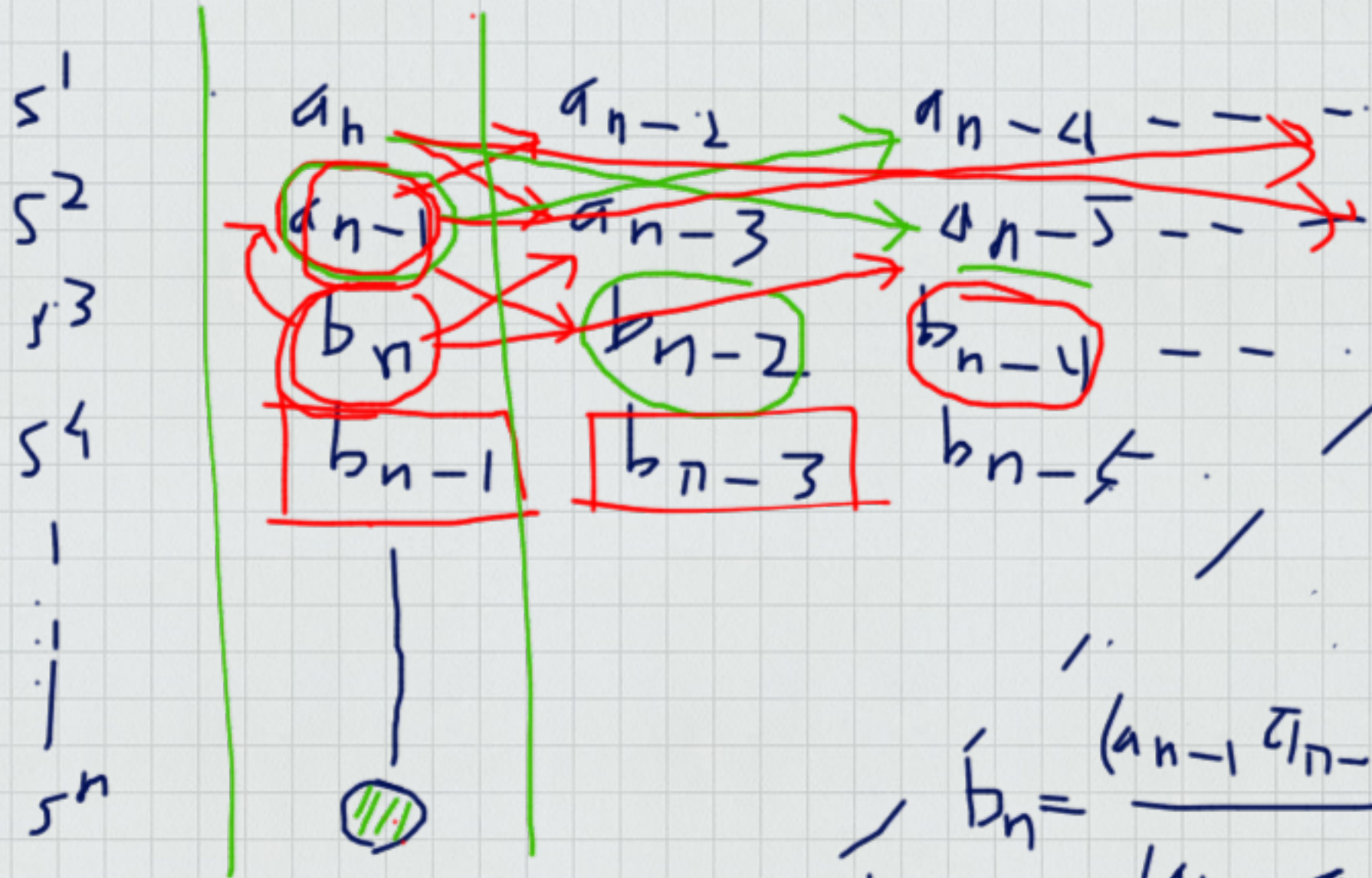
Suatu persamaan karakteristik dari sistem kendali yang dikategorikan sebagai LTI, semua akarnya akan berada di sebelah kiri sumbu khayal pada bidang kompleks (mempunyai bagian $\text{Re} < 0$), yang berarti sistem kendali itu stabil, jika polynomial $P(s)$ -nya memenuhi 2 (dua) kriteria Routh:

- (1) Semua koefisien $P(s)$ $a_0, \dots, a_n > 0$
- (2) Tidak ada pergantian tanda pada kolom pertama Tabel Routh.

Jika kriteria pertama tidak terpenuhi,
TIDAK PERLU lagi memeriksa kriteria kedua.

$$P(s) = a_n s^n + a_{n-1} s^{n-1} + a_{n-2} s^{n-2} + \dots + a_1 s + a_0$$

Tabel Routh =



Kolom Pertama

$$b_n = \frac{(a_{n-1} a_{n-2}) - (a_n a_{n-3})}{a_{n-1}}$$

$$b_{n-2} = \frac{(a_{n-1} a_{n-4}) - (a_n a_{n-5})}{a_{n-1}}$$

Contoh:

* $P(s) = s^5 + 3s^3 + 2s^2 + s + 1 \rightarrow$ tidak stabil

* $P(s) = -s^2 - 2s - 3 \Rightarrow P(s) = s^2 + 2s + 3 = 0 \rightarrow$
stabil

* $P(s) = s^5 + 4s^4 + 3s^3 + 2s^2 + s + 1$

Kriteria (1) terpenuhi

Kriteria (2):

Tabel Routh:

terjadi
pergantian
tanda, sistem
tidak stabil

s^1	1	3	1
s^2	4	2	1
s^3	2,5	0,75	0
s^4	0,8	1	0
s^5	-2,375	0	0
s^6	1	0	0

$$* P(s) = s^5 + 4s^4 + 3s^3 + 12s^2 + 5s + 1$$

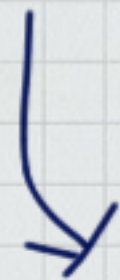
Kriteria (1) Terpenuhi

Kriteria (2) → Tabel Routh

$$\begin{array}{r|rrrr} s^1 & 1 & 3 & 1 & \\ s^2 & 4 & 12 & 1 & \end{array}$$

$$\begin{array}{r|rrrr} s^3 & \epsilon & 0,75 & 0 & \end{array}$$

$$\begin{array}{r|rrrr} s^4 & \frac{12\epsilon - 3}{\epsilon} & 1 & 0 & \end{array}$$



Pergantian

tanda → tidak stabil.

$\epsilon = \text{epsilon} > 0$
(sangat kecil)

$$\frac{12\epsilon - 3}{\epsilon} = 12 - \frac{3}{\epsilon} < 0$$

* Suatu sistem kendali dengan Kompensator memiliki spesifikasi: $G(s) = \frac{10}{s^2}$ (kendalian double integrator) dan $H(s) = \frac{K(s+1)}{(s+2)}$ (lag Kompensator).
 Tentukan batas K supaya stabil.

$$* P(s) = s^5 + 4s^4 + 3s^3 + 12s^2 + 5s + 1$$

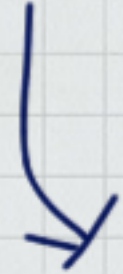
Kriteria (1) Terpenuhi

Kriteria (2) → Tabel Routh

$$\begin{array}{r|rrrr} s^1 & 1 & 3 & 1 & \\ s^2 & 4 & 12 & 1 & \end{array}$$

$$\begin{array}{r|rrrr} s^3 & \epsilon & 0,75 & 0 & \end{array}$$

$$\begin{array}{r|rrrr} s^4 & \frac{12\epsilon - 3}{\epsilon} & 1 & 0 & \end{array}$$



Pergantian
tanda → tidak stabil.

$\epsilon = \text{epsilon} > 0$
(sangat kecil)

$$\frac{12\epsilon - 3}{\epsilon} = 12 - \frac{3}{\epsilon} < 0$$

* Suatu sistem kendali dengan Kompensator memiliki spesifikasi: $G(s) = \frac{10}{s^2}$ (kendalian double integrator) dan $H(s) = \frac{K(s+1)}{(s+2)}$ (lag Kompensator). Tentukan batas K supaya stabil!

Pers. Karakteristik : $1 + Z(s)H(s) = 0$

$$1 + \left(\frac{10}{s^2}\right) \left(\frac{K(s+1)}{(s+4)}\right) = 0$$

$$s^2(s+1) + 10Ks + 10K = 0$$

$$s^3 + 2s^2 + 10Ks + 10K = 0$$

* Kriteria (1) $\rightarrow K > 0$

* Kriteria (2) Tabel Routh

s^1	1	$10K$
s^2	2	$10K$
s^3	$5K$	0
s^4	$10K$	0

$$5K > 0 \rightarrow K > 0$$

Coba untuk
 $H(s) = K \frac{s+2}{s+1}$
 Phase Lead
 Compensator

Jadi sistem Kendali akan stabil asal $K > 0$

Chapter 4

The Laplace Transform

4.1 INTRODUCTION

Several techniques used in solving engineering problems are based on the replacement of functions of a real variable (usually time or distance) by certain frequency dependent representations, or by functions of a complex variable dependent upon frequency. A typical example is the use of Fourier series to solve certain electrical problems. One such problem consists of finding the current in some part of a linear electrical network in which the input voltage is a periodic or repeating waveform. The periodic voltage may be replaced by its Fourier series representation, and the current produced by each term of the series can then be determined. The total current is the sum of the individual currents (superposition). This technique often results in a substantial savings in computational effort.

A transformation technique relating time functions to frequency dependent functions of a complex variable is presented in the next few sections of this chapter. It is called the *Laplace transform*. The application of this mathematical transformation to solving linear constant coefficient differential equations is discussed in the remaining sections, and provides the basis for the analysis and design techniques developed in subsequent chapters.

4.2 THE LAPLACE TRANSFORM

The Laplace transform is defined in the following manner:

Definition 4.1: Let $f(t)$ be a real function of a real variable t defined for $t > 0$. Then

$$\mathcal{L}[f(t)] \equiv F(s) \equiv \lim_{\substack{T \rightarrow \infty \\ \epsilon \rightarrow 0}} \int_{\epsilon}^T f(t)e^{-st} dt = \int_{0^+}^{\infty} f(t)e^{-st} dt, \quad 0 < \epsilon < T$$

is called the **Laplace transform** of $f(t)$. s is a complex variable defined by $s = \sigma + j\omega$, where σ and ω are real variables* and $j = \sqrt{-1}$.

Note that the lower limit on the integral is $t = \epsilon > 0$. This definition of the lower limit is sometimes useful in dealing with functions which are discontinuous at $t = 0$. When *explicit* use is made of this limit, it will be abbreviated $t = \lim_{\epsilon \rightarrow 0} \epsilon \equiv 0^+$, as shown above in the integral on the right.

The real variable t always denotes *time*.

* The real part σ of a complex variable s is often written as $\text{Re}(s)$ (the real part of s) and the imaginary part ω as $\text{Im}(s)$ (the imaginary part of s). Parentheses are placed around s only when there is a possibility of confusion.

Definition 4.2: If $f(t)$ is defined and single-valued for $t > 0$ and $F(\sigma)$ is absolutely convergent for some real number σ_0 , that is,

$$\int_{0+}^{\infty} |f(t)| e^{-\sigma_0 t} dt = \lim_{\substack{T \rightarrow \infty \\ \epsilon \rightarrow 0}} \int_{\epsilon}^T |f(t)| e^{-\sigma_0 t} dt < +\infty, \quad 0 < \epsilon < T$$

then $f(t)$ is **Laplace transformable** for $\text{Re}(s) > \sigma_0$.

Example 4.1.

The function e^{-t} is Laplace transformable since

$$\int_{0+}^{\infty} |e^{-t}| e^{-\sigma_0 t} dt = \int_{0+}^{\infty} e^{-(1+\sigma_0)t} dt = \frac{1}{-(1+\sigma_0)} e^{-(1+\sigma_0)t} \Big|_{0+}^{\infty} = \frac{1}{1+\sigma_0} < +\infty$$

if $1+\sigma_0 > 0$ or $\sigma_0 > -1$.

Example 4.2.

The Laplace transform of e^{-t} is

$$\mathcal{L}[e^{-t}] = \int_{0+}^{\infty} e^{-t} e^{-st} dt = \frac{-1}{(s+1)} e^{-(s+1)t} \Big|_{0+}^{\infty} = \frac{1}{s+1} \quad \text{for } \text{Re}(s) > -1$$

4.3 THE INVERSE LAPLACE TRANSFORM

The Laplace transform transforms a problem from the real variable time domain into the complex variable s domain. After a solution of the transformed problem has been obtained in terms of s , it is necessary to "invert" this transform in order to obtain the time domain solution. The transformation from the s domain into the t domain is called the *inverse Laplace transform*.

Definition 4.3: Let $F(s)$ be the Laplace transform of a function $f(t)$, $t > 0$. The contour integral

$$\mathcal{L}^{-1}[F(s)] \equiv f(t) = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F(s) e^{st} ds$$

where $j = \sqrt{-1}$ and $c > \sigma_0$ (σ_0 as given in Definition 4.2) is called the **inverse Laplace transform** of $F(s)$.

It is seldom necessary in practice to perform the contour integration defined in Definition 4.3. For applications of the Laplace transform in this book, it is never necessary. A simple technique for evaluating the inverse transform for most control system problems is presented in Section 4.8.

4.4 SOME PROPERTIES OF THE LAPLACE TRANSFORM AND ITS INVERSE

The Laplace transform and its inverse have several important properties which can be used advantageously in the solution of linear constant coefficient differential equations. They are:

1. The Laplace transform is a *linear transformation* between functions defined in the t domain and functions defined in the s domain. That is, if $F_1(s)$ and $F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, then $a_1 F_1(s) + a_2 F_2(s)$ is the Laplace transform of $a_1 f_1(t) + a_2 f_2(t)$, where a_1 and a_2 are arbitrary constants.
2. The inverse Laplace transform is a *linear transformation* between functions defined in the s domain and functions defined in the t domain. That is, if $f_1(t)$ and $f_2(t)$ are the inverse Laplace transforms of $F_1(s)$ and $F_2(s)$, respectively, then $b_1 f_1(t) + b_2 f_2(t)$ is the inverse Laplace transform of $b_1 F_1(s) + b_2 F_2(s)$, where b_1 and b_2 are arbitrary constants.

3. The Laplace transform of the *derivative* df/dt of a function $f(t)$ whose Laplace transform is $F(s)$ is

$$\mathcal{L}[df/dt] = sF(s) - f(0^+)$$

where $f(0^+)$ is the initial value of $f(t)$, evaluated as the one-sided limit of $f(t)$ as t approaches zero from positive values.

4. The Laplace transform of the *integral* $\int_0^t f(\tau) d\tau$ of a function $f(t)$ whose Laplace transform is $F(s)$ is

$$\mathcal{L}\left[\int_0^t f(\tau) d\tau\right] = \frac{F(s)}{s}$$

5. The initial value $f(0^+)$ of the function $f(t)$ whose Laplace transform is $F(s)$ is

$$f(0^+) = \lim_{t \rightarrow 0} f(t) = \lim_{s \rightarrow \infty} sF(s) \quad t > 0$$

This relation is called the *Initial Value Theorem*.

6. The final value $f(\infty)$ of the function $f(t)$ whose Laplace transform is $F(s)$ is

$$f(\infty) = \lim_{t \rightarrow \infty} f(t) = \lim_{s \rightarrow 0} sF(s)$$

if $\lim_{t \rightarrow \infty} f(t)$ exists. This relation is called the *Final Value Theorem*.

7. The Laplace transform of a function $f(t/a)$ (*Time Scaling*) is

$$\mathcal{L}[f(t/a)] = aF(as)$$

where $F(s) = \mathcal{L}[f(t)]$.

8. The inverse Laplace transform of the function $F(s/a)$ (*Frequency Scaling*) is

$$\mathcal{L}^{-1}[F(s/a)] = af(at)$$

where $\mathcal{L}^{-1}[F(s)] = f(t)$.

9. The Laplace transform of the function $f(t-T)$ (*Time Delay*) where $T > 0$ and $f(t-T) = 0$ for $t \leq T$, is

$$\mathcal{L}[f(t-T)] = e^{-sT} F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

10. The Laplace transform of the function $e^{-at}f(t)$ is given by

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

where $F(s) = \mathcal{L}[f(t)]$. (*Complex Translation*)

11. The Laplace transform of the *product of two functions* $f_1(t)$ and $f_2(t)$ is given by the *complex convolution integral*

$$\mathcal{L}[f_1(t) \cdot f_2(t)] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} F_1(\omega) F_2(s-\omega) d\omega$$

where $F_1(s) = \mathcal{L}[f_1(t)]$, $F_2(s) = \mathcal{L}[f_2(t)]$.

12. The inverse Laplace transform of the *product of the two transforms* $F_1(s)$ and $F_2(s)$ is given by the *convolution integrals*

$$\mathcal{L}^{-1}[F_1(s) \cdot F_2(s)] = \int_0^t f_1(\tau) f_2(t-\tau) d\tau = \int_0^t f_2(\tau) f_1(t-\tau) d\tau$$

where $\mathcal{L}^{-1}[F_1(s)] = f_1(t)$, $\mathcal{L}^{-1}[F_2(s)] = f_2(t)$.

Example 4.3.

The Laplace transforms of the functions e^{-t} and e^{-2t} are $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$, $\mathcal{L}[e^{-2t}] = \frac{1}{s+2}$. Then by Property 1,

$$\mathcal{L}[3e^{-t} - e^{-2t}] = 3\mathcal{L}[e^{-t}] - \mathcal{L}[e^{-2t}] = \frac{3}{s+1} - \frac{1}{s+2} = \frac{2s+5}{s^2+3s+2}$$

Example 4.4.

The inverse Laplace transforms of the functions $\frac{1}{s+1}$ and $\frac{1}{s+3}$ are

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}, \quad \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$

Then by Property 2,

$$\mathcal{L}^{-1}\left[\frac{2}{s+1} - \frac{4}{s+3}\right] = 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = 2e^{-t} - 4e^{-3t}$$

Example 4.5.

The Laplace transform of $\frac{d}{dt}(e^{-t})$ can be determined by application of Property 3. Since $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$ and $\lim_{t \rightarrow 0} e^{-t} = 1$, then

$$\mathcal{L}\left[\frac{d}{dt}(e^{-t})\right] = s\left(\frac{1}{s+1}\right) - 1 = \frac{-1}{s+1}$$

Example 4.6.

The Laplace transform of $\int_0^t e^{-\tau} d\tau$ can be determined by application of Property 4. Since $\mathcal{L}[e^{-t}] = \frac{1}{s+1}$, then

$$\mathcal{L}\left[\int_0^t e^{-\tau} d\tau\right] = \frac{1}{s}\left(\frac{1}{s+1}\right) = \frac{1}{s(s+1)}$$

Example 4.7.

The Laplace transform of e^{-3t} is $\mathcal{L}[e^{-3t}] = \frac{1}{s+3}$. The initial value of e^{-3t} can be determined by the Initial Value Theorem as $\lim_{t \rightarrow 0} e^{-3t} = \lim_{s \rightarrow \infty} s\left(\frac{1}{s+3}\right) = 1$.

Example 4.8.

The Laplace transform of the function $(1 - e^{-t})$ is $\frac{1}{s(s+1)}$. The final value of this function can be determined from the Final Value Theorem as $\lim_{t \rightarrow \infty} (1 - e^{-t}) = \lim_{s \rightarrow 0} \frac{s}{s(s+1)} = 1$.

Example 4.9.

The Laplace transform of e^{-t} is $\frac{1}{s+1}$. The Laplace transform of e^{-3t} can be determined by application of Property 7 (Time Scaling), where $a = \frac{1}{3}$: $\mathcal{L}[e^{-3t}] = \frac{1}{3}\left[\frac{1}{(\frac{1}{3}s+1)}\right] = \frac{1}{s+3}$.

Example 4.10.

The inverse transform of $\frac{1}{s+1}$ is e^{-t} . The inverse transform of $\frac{1}{\frac{1}{3}s+1}$ can be determined by application of Property 8 (Frequency Scaling): $\mathcal{L}^{-1}\left[\frac{1}{\frac{1}{3}s+1}\right] = 3e^{-3t}$.

Example 4.11.

The Laplace transform of the function e^{-t} is $\frac{1}{s+1}$. The Laplace transform of the function defined as

$$f(t) = \begin{cases} e^{-(t-2)} & t > 2 \\ 0 & t \leq 2 \end{cases}$$

can be determined by Property 9, with $T=2$: $\mathcal{L}[f(t)] = e^{-2s} \cdot \mathcal{L}[e^{-t}] = \frac{e^{-2s}}{s+1}$.

Example 4.12.

The Laplace transform of $\cos t$ is $\frac{s}{s^2 + 1}$. The Laplace transform of $e^{-2t} \cos t$ can be determined from Property 10 with $a = 2$: $\mathcal{L}[e^{-2t} \cos t] = \frac{s + 2}{(s + 2)^2 + 1} = \frac{s + 2}{s^2 + 4s + 5}$.

Example 4.13.

The Laplace transform of the product $e^{-2t} \cos t$ can be determined by application of Property 11 (Complex Convolution). That is, since $\mathcal{L}[e^{-2t}] = \frac{1}{s + 2}$ and $\mathcal{L}[\cos t] = \frac{s}{s^2 + 1}$, then

$$\mathcal{L}[e^{-2t} \cos t] = \frac{1}{2\pi j} \int_{c-j\infty}^{c+j\infty} \left(\frac{\omega}{\omega^2 + 1} \right) \left(\frac{1}{s - \omega + 2} \right) d\omega = \frac{s + 2}{s^2 + 4s + 5}$$

The details of this contour integration are not carried out here because they are too complicated (see, for example, Reference [5]) and unnecessary. The Laplace transform of $e^{-2t} \cos t$ was very simply determined in Example 4.12 using Property 10. There are, however, many instances in more advanced treatments of automatic control in which complex convolution can be used effectively.

Example 4.14.

The inverse Laplace transform of the function $F(s) = \frac{s}{(s + 1)(s^2 + 1)}$ can be determined by application of Property 12. Since $\mathcal{L}^{-1}\left[\frac{1}{s + 1}\right] = e^{-t}$ and $\mathcal{L}^{-1}\left[\frac{s}{s^2 + 1}\right] = \cos t$, then

$$\mathcal{L}^{-1}\left[\left(\frac{1}{s + 1}\right)\left(\frac{s}{s^2 + 1}\right)\right] = \int_{0+}^t e^{-(t-\tau)} \cos \tau d\tau = e^{-t} \int_{0+}^t e^{\tau} \cos \tau d\tau = \frac{1}{2}(\cos t + \sin t - e^{-t})$$

4.5 SHORT TABLE OF LAPLACE TRANSFORMS

The following is a short table of Laplace transforms. It is not complete, but when used in conjunction with the properties of the Laplace transform described in Section 4.4 and the partial fraction expansion techniques described in Section 4.7, it is adequate to handle all of the problems in this book. A more complete table of Laplace transform pairs is found in the Appendix.

TABLE 4.1

Time Function		Laplace Transform
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Unit Ramp	t	$1/s^2$
Polynomial	t^n	$n!/s^{n+1}$
Exponential	e^{-at}	$\frac{1}{s + a}$
Sine Wave	$\sin \omega t$	$\frac{\omega}{s^2 + \omega^2}$
Cosine Wave	$\cos \omega t$	$\frac{s}{s^2 + \omega^2}$
Damped Sine Wave	$e^{-at} \sin \omega t$	$\frac{\omega}{(s + a)^2 + \omega^2}$
Damped Cosine Wave	$e^{-at} \cos \omega t$	$\frac{s + a}{(s + a)^2 + \omega^2}$

and $z_2 = -1$.

Example

Appendix

SOME LAPLACE TRANSFORM PAIRS USEFUL
FOR CONTROL SYSTEMS ANALYSIS

$F(s)$	$f(t)$	$t > 0$
1	$\delta(t)$	unit impulse
e^{-Ts}	$\delta(t - T)$	delayed impulse
$\frac{1}{s + a}$	e^{-at}	
$\frac{1}{(s + a)^n}$	$\frac{1}{(n-1)!} t^{n-1} e^{-at}$	$n = 1, 2, 3, \dots$
$\frac{1}{(s + a)(s + b)}$	$\frac{1}{b - a} (e^{-at} - e^{-bt})$	
$\frac{s}{(s + a)(s + b)}$	$\frac{1}{a - b} (ae^{-at} - be^{-bt})$	
$\frac{s + z}{(s + a)(s + b)}$	$\frac{1}{b - a} [(z - a)e^{-at} - (z - b)e^{-bt}]$	
$\frac{1}{(s + a)(s + b)(s + c)}$	$\frac{e^{-at}}{(b - a)(c - a)} + \frac{e^{-bt}}{(c - b)(a - b)} + \frac{e^{-ct}}{(a - c)(b - c)}$	
$\frac{s + z}{(s + a)(s + b)(s + c)}$	$\frac{(z - a)e^{-at}}{(b - a)(c - a)} + \frac{(z - b)e^{-bt}}{(c - b)(a - b)} + \frac{(z - c)e^{-ct}}{(a - c)(b - c)}$	
$\frac{\omega}{s^2 + \omega^2}$	$\sin \omega t$	
$\frac{s}{s^2 + \omega^2}$	$\cos \omega t$	
$\frac{s + z}{s^2 + \omega^2}$	$\sqrt{\frac{z^2 + \omega^2}{\omega^2}} \sin(\omega t + \phi)$	$\phi \equiv \tan^{-1}(\omega/z)$
$\frac{s \sin \phi + \omega \cos \phi}{s^2 + \omega^2}$	$\sin(\omega t + \phi)$	
$\frac{1}{(s + a)^2 + \omega^2}$	$\frac{1}{\omega} e^{-at} \sin \omega t$	

$F(s)$	$f(t)$ $t > 0$
$\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$	$\frac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \quad \omega_d \equiv \omega_n \sqrt{1 - \zeta^2}$
$\frac{s + a}{(s + a)^2 + \omega^2}$	$e^{-at} \cos \omega t$
$\frac{s + z}{(s + a)^2 + \omega^2}$	$\sqrt{\frac{(z - a)^2 + \omega^2}{\omega^2}} e^{-at} \sin(\omega t + \phi) \quad \phi \equiv \tan^{-1} \left(\frac{\omega}{z - a} \right)$
$\frac{1}{s}$	$u(t)$ or 1 unit step
$\frac{1}{s} e^{-Ts}$	$u(t - T)$ delayed step
$\frac{1}{s} (1 - e^{-Ts})$	$u(t) - u(t - T)$ rectangular pulse
$\frac{1}{s(s + a)}$	$\frac{1}{a} (1 - e^{-at})$
$\frac{1}{s(s + a)(s + b)}$	$\frac{1}{ab} \left(1 - \frac{be^{-at}}{b - a} + \frac{ae^{-bt}}{b - a} \right)$
$\frac{s + z}{s(s + a)(s + b)}$	$\frac{1}{ab} \left(z - \frac{b(z - a)e^{-at}}{b - a} + \frac{a(z - b)e^{-bt}}{b - a} \right)$
$\frac{1}{s(s^2 + \omega^2)}$	$\frac{1}{\omega^2} (1 - \cos \omega t)$
$\frac{s + z}{s(s^2 + \omega^2)}$	$\frac{z}{\omega^2} - \sqrt{\frac{z^2 + \omega^2}{\omega^4}} \cos(\omega t + \phi) \quad \phi \equiv \tan^{-1}(\omega/z)$
$\frac{1}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$	$\frac{1}{\omega_n^2} - \frac{1}{\omega_n \omega_d} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi)$ $\omega_d \equiv \omega_n \sqrt{1 - \zeta^2} \quad \phi \equiv \cos^{-1} \zeta$
$\frac{1}{s(s + a)^2}$	$\frac{1}{a^2} (1 - e^{-at} - ate^{-at})$
$\frac{s + z}{s(s + a)^2}$	$\frac{1}{a^2} [z - ze^{-at} + a(a - z)te^{-at}]$
$\frac{1}{s^2}$	t unit ramp
$\frac{1}{s^2(s + a)}$	$\frac{1}{a^2} (at - 1 + e^{-at})$
$\frac{1}{s^n} \quad n = 1, 2, 3, \dots$	$\frac{t^{n-1}}{(n-1)!} \quad 0! = 1$

1. Zadeh
McGr2. Hartl
the Li3. Bliss,
Soc. A4. Reicha
(Germ5. McLac
Unive6. Church
New Y7. Desoer
on Cir

8. Spiege

9. Krall,
Society
pp. 6410. Elgerd
Zero L
actions

B.2 TABLE OF z -TRANSFORMS

$\mathcal{F}(s)$ is the Laplace transform of $f(t)$ and $F(z)$ is the z -transform of $f(nT)$. Unless otherwise noted, $f(t) = 0$, $t < 0$ and the region of convergence of $F(z)$ is outside a circle $r < |z|$ such that all poles of $F(z)$ are inside r .

Table B.2

Number	$\mathcal{F}(s)$	$f(nT)$	$F(z)$
1	—	$1, n = 0; 0 \text{ } n \neq 0$	1
2	—	$1, n = k; 0 \text{ } n \neq k$	z^{-k}
3	$\frac{1}{s}$	$1(nT)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	nT	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(nT)^2$	$\frac{T^2 z(z+1)}{2(z-1)^3}$
6	$\frac{1}{s^4}$	$\frac{1}{3!}(nT)^3$	$\frac{T^3 z(z^2+4z+1)}{6(z-1)^4}$
7	$\frac{1}{s^m}$	$\lim_{a \rightarrow 0} \frac{(-1)^{m-1}}{(m-1)!} \frac{\partial^{m-1}}{\partial a^{m-1}} e^{-anT}$	$\lim_{a \rightarrow 0} \frac{(-1)^{m-1}}{(m-1)!} \frac{\partial^{m-1}}{\partial a^{m-1}} \frac{z}{z - e^{-aT}}$
8	$\frac{1}{s+a}$	e^{-anT}	$\frac{z}{z - e^{-aT}}$
9	$\frac{1}{(s+a)^2}$	nTe^{-anT}	$\frac{Tze^{-aT}}{(z - e^{-aT})^2}$
10	$\frac{1}{(s+a)^3}$	$\frac{1}{2}(nT)^2 e^{-anT}$	$\frac{T^2 e^{-aT} z(z + e^{-aT})}{2(z - e^{-aT})^3}$
11	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!} \frac{\partial^{m-1}}{\partial a^{m-1}} (e^{-anT})$	$\frac{(-1)^{m-1}}{(m-1)!} \frac{\partial^{m-1}}{\partial a^{m-1}} \frac{z}{z - e^{-aT}}$
12	$\frac{a}{s(s+a)}$	$1 - e^{-anT}$	$\frac{z(1 - e^{-aT})}{(z-1)(z - e^{-aT})}$

Number	$\mathcal{F}(s)$	$f(nT)$	$F(z)$
13	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(anT - 1 + e^{-anT})$	$\frac{z[(aT - 1 + e^{-aT})z + (1 - e^{-aT} - aTe^{-aT})]}{a(z-1)^2(z - e^{-aT})}$
14	$\frac{b-a}{(s+a)(s+b)}$	$(e^{-anT} - e^{-bnT})$	$\frac{(e^{-aT} - e^{-bT})z}{(z - e^{-aT})(z - e^{-bT})}$
15	$\frac{s}{(s+a)^2}$	$(1 - anT)e^{-anT}$	$\frac{z[z - e^{-aT}(1 + aT)]}{(z - e^{-aT})^2}$
16	$\frac{a^2}{s(s+a)^2}$	$1 - e^{-anT}(1 + anT)$	$\frac{z[z(1 - e^{-aT} - aTe^{-aT}) + e^{-2aT} - e^{-aT} + aTe^{-aT}]}{(z-1)(z - e^{-aT})^2}$
17	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bnT} - ae^{-anT}$	$\frac{z[z(b-a) - (be^{-aT} - ae^{-bT})]}{(z - e^{-aT})(z - e^{-bT})}$
18	$\frac{a}{s^2 + a^2}$	$\sin anT$	$\frac{z \sin aT}{z^2 - (2 \cos aT)z + 1}$
19	$\frac{s}{s^2 + a^2}$	$\cos anT$	$\frac{z(z - \cos aT)}{z^2 - (2 \cos aT)z + 1}$
20	$\frac{s+a}{(s+a)^2 + b^2}$	$e^{-anT} \cos bnT$	$\frac{z(z - e^{-aT} \cos bT)}{z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT}}$
21	$\frac{b}{(s+a)^2 + b^2}$	$e^{-anT} \sin bnT$	$\frac{ze^{-aT} \sin bT}{z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT}}$
22	$\frac{a^2 + b^2}{s((s+a)^2 + b^2)}$	$1 - e^{-anT} \left(\cos bnT + \frac{a}{b} \sin bnT \right)$	$\frac{z(Az + B)}{(z-1)(z^2 - 2e^{-aT}(\cos bT)z + e^{-2aT})}$ $A = 1 - e^{-aT} \cos bT - \frac{a}{b} e^{-aT} \sin bT$ $B = e^{-2aT} + \frac{a}{b} e^{-aT} \sin bT - e^{-aT} \cos bT$