Chapter 4

The Laplace Transform

4.1 INTRODUCTION

Several techniques used in solving engineering problems are based on the replacement of functions of a real variable (usually time or distance) by certain frequency dependent representations, or by functions of a complex variable dependent upon frequency. A typical example is the use of Fourier series to solve certain electrical problems. One such problem consists of finding the current in some part of a linear electrical network in which the input voltage is a periodic or repeating waveform. The periodic voltage may be replaced by its Fourier series representation, and the current produced by each term of the series can then be determined. The total current is the sum of the individual currents (superposition). This technique often results in a substantial savings in computational effort.

A transformation technique relating time functions to frequency dependent functions of a complex variable is presented in the next few sections of this chapter. It is called the *Laplace transform*. The application of this mathematical transformation to solving linear constant coefficient differential equations is discussed in the remaining sections, and provides the basis for the analysis and design techniques developed in subsequent chapters.

4.2 THE LAPLACE TRANSFORM

The Laplace transform is defined in the following manner:

Definition 4.1: Let $f(t)$ be a real function of a real variable t defined for $t > 0$. Then

$$\mathcal{L}[f(t)] \;\equiv\; F(s) \;\equiv\; \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \int_{\epsilon}^{T} f(t)e^{-st}\,dt \;=\; \int_{0^+}^{\infty} f(t)e^{-st}\,dt, \qquad 0<\epsilon<T$$

is called the **Laplace transform** of $f(t)$. s is a complex variable defined by $s \equiv \sigma + j\omega$, where σ and ω are real variables* and $j = \sqrt{-1}$.

Note that the lower limit on the integral is $t = \epsilon > 0$. This definition of the lower limit is sometimes useful in dealing with functions which are discontinuous at $t = 0$. When *explicit* use is made of this limit, it will be abbreviated $t = \lim_{\epsilon\to 0} \epsilon \equiv 0^+$, as shown above in the integral on the right.

The real variable t always denotes *time*.

* The real part σ of a complex variable s is often written as Re(s) (the real part of s) and the imaginary part ω as Im(s) (the imaginary part of s). Parentheses are placed around s only when there is a possibility of confusion.

Definition 4.2: If $f(t)$ is defined and single-valued for $t>0$ and $F(\sigma)$ is absolutely convergent for some real number σ_0, that is,

$$\int_{0^+}^{\infty} |f(t)|\, e^{-\sigma_0 t}\, dt \quad = \quad \lim_{\substack{T\to\infty \\ \epsilon\to 0}} \int_{\epsilon}^{T} |f(t)|\, e^{-\sigma_0 t}\, dt \ < \ +\infty, \qquad 0<\epsilon<T$$

then $f(t)$ is **Laplace transformable** for $\text{Re}(s) > \sigma_0$.

Example 4.1.

The function e^{-t} is Laplace transformable since

$$\int_{0^+}^{\infty} |e^{-t}|\, e^{-\sigma_0 t}\, dt \quad = \quad \int_{0^+}^{\infty} e^{-(1+\sigma_0)t}\, dt \quad = \quad \frac{1}{-(1+\sigma_0)}\, e^{-(1+\sigma_0)t}\bigg|_{0^+}^{\infty} \quad = \quad \frac{1}{1+\sigma_0} \ < \ +\infty$$

if $1+\sigma_0 > 0$ or $\sigma_0 > -1$.

Example 4.2.

The Laplace transform of e^{-t} is

$$\mathcal{L}[e^{-t}] \quad = \quad \int_{0^+}^{\infty} e^{-t}\, e^{-st}\, dt \quad = \quad \frac{-1}{(s+1)}\, e^{-(s+1)t}\bigg|_{0^+}^{\infty} \quad = \quad \frac{1}{s+1} \qquad \text{for } \text{Re}(s) > -1$$

4.3 THE INVERSE LAPLACE TRANSFORM

The Laplace transform transforms a problem from the real variable time domain into the complex variable s domain. After a solution of the transformed problem has been obtained in terms of s, it is necessary to "invert" this transform in order to obtain the time domain solution. The transformation from the s domain into the t domain is called the *inverse Laplace transform.*

Definition 4.3: Let $F(s)$ be the Laplace transform of a function $f(t)$, $t>0$. The contour integral

$$\mathcal{L}^{-1}[F(s)] \quad \equiv \quad f(t) \quad = \quad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F(s)\, e^{st}\, ds$$

where $j=\sqrt{-1}$ and $c > \sigma_0$ (σ_0 as given in Definition 4.2) is called the **inverse Laplace transform** of $F(s)$.

It is seldom necessary in practice to perform the contour integration defined in Definition 4.3. For applications of the Laplace transform in this book, it is never necessary. A simple technique for evaluating the inverse transform for most control system problems is presented in Section 4.8.

4.4 SOME PROPERTIES OF THE LAPLACE TRANSFORM AND ITS INVERSE

The Laplace transform and its inverse have several important properties which can be used advantageously in the solution of linear constant coefficient differential equations. They are:

1. The Laplace transform is a *linear transformation* between functions defined in the t domain and functions defined in the s domain. That is, if $F_1(s)$ and $F_2(s)$ are the Laplace transforms of $f_1(t)$ and $f_2(t)$, respectively, then $a_1F_1(s) + a_2F_2(s)$ is the Laplace transform of $a_1f_1(t) + a_2f_2(t)$, where a_1 and a_2 are arbitrary constants.

2. The inverse Laplace transform is a *linear transformation* between functions defined in the s domain and functions defined in the t domain. That is, if $f_1(t)$ and $f_2(t)$ are the inverse Laplace transforms of $F_1(s)$ and $F_2(s)$, respectively, then $b_1f_1(t) + b_2f_2(t)$ is the inverse Laplace transform of $b_1F_1(s) + b_2F_2(s)$, where b_1 and b_2 are arbitrary constants.

3. The Laplace transform of the *derivative* df/dt of a function $f(t)$ whose Laplace transform is $F(s)$ is

$$\mathcal{L}[df/dt] = sF(s) - f(0^+)$$

where $f(0^+)$ is the initial value of $f(t)$, evaluated as the one-sided limit of $f(t)$ as t approaches zero from positive values.

4. The Laplace transform of the *integral* $\int_0^t f(\tau)\,d\tau$ of a function $f(t)$ whose Laplace transform is $F(s)$ is

$$\mathcal{L}\left[\int_0^t f(\tau)\,d\tau\right] = \frac{F(s)}{s}$$

5. The initial value $f(0^+)$ of the function $f(t)$ whose Laplace transform is $F(s)$ is

$$f(0^+) = \lim_{t\to 0} f(t) = \lim_{s\to\infty} sF(s) \qquad t > 0$$

This relation is called the *Initial Value Theorem.*

6. The final value $f(\infty)$ of the function $f(t)$ whose Laplace transform is $F(s)$ is

$$f(\infty) = \lim_{t\to\infty} f(t) = \lim_{s\to 0} sF(s)$$

if $\lim_{t\to\infty} f(t)$ exists. This relation is called the *Final Value Theorem.*

7. The Laplace transform of a function $f(t/a)$ (*Time Scaling*) is

$$\mathcal{L}[f(t/a)] = aF(as)$$

where $F(s) = \mathcal{L}[f(t)]$.

8. The inverse Laplace transform of the function $F(s/a)$ (*Frequency Scaling*) is

$$\mathcal{L}^{-1}[F(s/a)] = af(at)$$

where $\mathcal{L}^{-1}[F(s)] = f(t)$.

9. The Laplace transform of the function $f(t-T)$ (*Time Delay*) where $T > 0$ and $f(t-T) = 0$ for $t \leqq T$, is

$$\mathcal{L}[f(t-T)] = e^{-sT}F(s)$$

where $F(s) = \mathcal{L}[f(t)]$.

10. The Laplace transform of the function $e^{-at}f(t)$ is given by

$$\mathcal{L}[e^{-at}f(t)] = F(s+a)$$

where $F(s) = \mathcal{L}[f(t)]$. (*Complex Translation*)

11. The Laplace transform of the *product of two functions* $f_1(t)$ and $f_2(t)$ is given by the *complex convolution integral*

$$\mathcal{L}[f_1(t)\cdot f_2(t)] = \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty} F_1(\omega)\,F_2(s-\omega)\,d\omega$$

where $F_1(s) = \mathcal{L}[f_1(t)]$, $F_2(s) = \mathcal{L}[f_2(t)]$.

12. The inverse Laplace transform of the *product of the two transforms* $F_1(s)$ and $F_2(s)$ is given by the *convolution integrals*

$$\mathcal{L}^{-1}[F_1(s)\cdot F_2(s)] = \int_{0^+}^{t} f_1(\tau)\,f_2(t-\tau)\,d\tau = \int_{0^+}^{t} f_2(\tau)\,f_1(t-\tau)\,d\tau$$

where $\mathcal{L}^{-1}[F_1(s)] = f_1(t)$, $\mathcal{L}^{-1}[F_2(s)] = f_2(t)$.

Example 4.3.

The Laplace transforms of the functions e^{-t} and e^{-2t} are $\mathcal{L}[e^{-t}] = \dfrac{1}{s+1}$, $\mathcal{L}[e^{-2t}] = \dfrac{1}{s+2}$. Then by Property 1,

$$\mathcal{L}[3e^{-t} - e^{-2t}] \;=\; 3\mathcal{L}[e^{-t}] - \mathcal{L}[e^{-2t}] \;=\; \frac{3}{s+1} - \frac{1}{s+2} \;=\; \frac{2s+5}{s^2+3s+2}$$

Example 4.4.

The inverse Laplace transforms of the functions $\dfrac{1}{s+1}$ and $\dfrac{1}{s+3}$ are

$$\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] = e^{-t}, \qquad \mathcal{L}^{-1}\left[\frac{1}{s+3}\right] = e^{-3t}$$

Then by Property 2,

$$\mathcal{L}^{-1}\left[\frac{2}{s+1} - \frac{4}{s+3}\right] \;=\; 2\mathcal{L}^{-1}\left[\frac{1}{s+1}\right] - 4\mathcal{L}^{-1}\left[\frac{1}{s+3}\right] \;=\; 2e^{-t} - 4e^{-3t}$$

Example 4.5.

The Laplace transform of $\dfrac{d}{dt}(e^{-t})$ can be determined by application of Property 3. Since $\mathcal{L}[e^{-t}] = \dfrac{1}{s+1}$ and $\lim_{t\to 0} e^{-t} = 1$, then

$$\mathcal{L}\left[\frac{d}{dt}(e^{-t})\right] \;=\; s\left(\frac{1}{s+1}\right) - 1 \;=\; \frac{-1}{s+1}$$

Example 4.6.

The Laplace transform of $\displaystyle\int_0^t e^{-\tau}\,d\tau$ can be determined by application of Property 4. Since $\mathcal{L}[e^{-t}] = \dfrac{1}{s+1}$, then

$$\mathcal{L}\left[\int_0^t e^{-\tau}\,d\tau\right] \;=\; \frac{1}{s}\left(\frac{1}{s+1}\right) \;=\; \frac{1}{s(s+1)}$$

Example 4.7.

The Laplace transform of e^{-3t} is $\mathcal{L}[e^{-3t}] = \dfrac{1}{s+3}$. The initial value of e^{-3t} can be determined by the Initial Value Theorem as $\lim_{t\to 0} e^{-3t} = \lim_{s\to\infty} s\left(\dfrac{1}{s+3}\right) = 1$.

Example 4.8.

The Laplace transform of the function $(1-e^{-t})$ is $\dfrac{1}{s(s+1)}$. The final value of this function can be determined from the Final Value Theorem as $\lim_{t\to\infty}(1-e^{-t}) = \lim_{s\to 0}\dfrac{s}{s(s+1)} = 1$.

Example 4.9.

The Laplace transform of e^{-t} is $\dfrac{1}{s+1}$. The Laplace transform of e^{-3t} can be determined by application of Property 7 (Time Scaling), where $a = \frac{1}{3}$: $\mathcal{L}[e^{-3t}] = \frac{1}{3}\left[\dfrac{1}{(\frac{1}{3}s+1)}\right] = \dfrac{1}{s+3}$.

Example 4.10.

The inverse transform of $\dfrac{1}{s+1}$ is e^{-t}. The inverse transform of $\dfrac{1}{\frac{1}{3}s+1}$ can be determined by application of Property 8 (Frequency Scaling): $\mathcal{L}^{-1}\left[\dfrac{1}{\frac{1}{3}s+1}\right] = 3e^{-3t}$

Example 4.11.

The Laplace transform of the function e^{-t} is $\dfrac{1}{s+1}$. The Laplace transform of the function defined as

$$f(t) \;=\; \begin{cases} e^{-(t-2)} & t > 2 \\ 0 & t \leq 2 \end{cases}$$

can be determined by Property 9, with $T = 2$: $\mathcal{L}[f(t)] = e^{-2s}\cdot\mathcal{L}[e^{-t}] = \dfrac{e^{-2s}}{s+1}$.

Example 4.12.

The Laplace transform of $\cos t$ is $\dfrac{s}{s^2+1}$. The Laplace transform of $e^{-2t}\cos t$ can be determined from Property 10 with $a=2$: $\mathcal{L}[e^{-2t}\cos t] = \dfrac{s+2}{(s+2)^2+1} = \dfrac{s+2}{s^2+4s+5}$.

Example 4.13.

The Laplace transform of the product $e^{-2t}\cos t$ can be determined by application of Property 11 (Complex Convolution). That is, since $\mathcal{L}[e^{-2t}] = \dfrac{1}{s+2}$ and $\mathcal{L}[\cos t] = \dfrac{s}{s^2+1}$, then

$$\mathcal{L}[e^{-2t}\cos t] \quad = \quad \frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}\left(\frac{\omega}{\omega^2+1}\right)\left(\frac{1}{s-\omega+2}\right)d\omega \quad = \quad \frac{s+2}{s^2+4s+5}$$

The details of this contour integration are not carried out here because they are too complicated (see, for example, Reference [5]) and unnecessary. The Laplace transform of $e^{-2t}\cos t$ was very simply determined in Example 4.12 using Property 10. There are, however, many instances in more advanced treatments of automatic control in which complex convolution can be used effectively.

Example 4.14.

The inverse Laplace transform of the function $F(s) = \dfrac{s}{(s+1)(s^2+1)}$ can be determined by application of Property 12. Since $\mathcal{L}^{-1}\left[\dfrac{1}{s+1}\right] = e^{-t}$ and $\mathcal{L}^{-1}\left[\dfrac{s}{s^2+1}\right] = \cos t$, then

$$\mathcal{L}^{-1}\left[\left(\frac{1}{s+1}\right)\left(\frac{s}{s^2+1}\right)\right] = \int_{0^+}^{t} e^{-(t-\tau)}\cos\tau\,d\tau = e^{-t}\int_{0^+}^{t} e^{\tau}\cos\tau\,d\tau = \tfrac{1}{2}(\cos t + \sin t - e^{-t})$$

4.5 SHORT TABLE OF LAPLACE TRANSFORMS

The following is a short table of Laplace transforms. It is not complete, but when used in conjunction with the properties of the Laplace transform described in Section 4.4 and the partial fraction expansion techniques described in Section 4.7, it is adequate to handle all of the problems in this book. A more complete table of Laplace transform pairs is found in the Appendix.

TABLE 4.1

Time Function		Laplace Transform
Unit Impulse	$\delta(t)$	1
Unit Step	$u(t)$	$1/s$
Unit Ramp	t	$1/s^2$
Polynomial	t^n	$n!/s^{n+1}$
Exponential	e^{-at}	$\dfrac{1}{s+a}$
Sine Wave	$\sin\omega t$	$\dfrac{\omega}{s^2+\omega^2}$
Cosine Wave	$\cos\omega t$	$\dfrac{s}{s^2+\omega^2}$
Damped Sine Wave	$e^{-at}\sin\omega t$	$\dfrac{\omega}{(s+a)^2+\omega^2}$
Damped Cosine Wave	$e^{-at}\cos\omega t$	$\dfrac{s+a}{(s+a)^2+\omega^2}$

Appendix

SOME LAPLACE TRANSFORM PAIRS USEFUL FOR CONTROL SYSTEMS ANALYSIS

$F(s)$	$f(t) \qquad t > 0$
1	$\delta(t)$ unit impulse
e^{-Ts}	$\delta(t-T)$ delayed impulse
$\dfrac{1}{s+a}$	e^{-at}
$\dfrac{1}{(s+a)^n}$	$\dfrac{1}{(n-1)!}t^{n-1}e^{-at} \qquad n = 1,2,3,\ldots$
$\dfrac{1}{(s+a)(s+b)}$	$\dfrac{1}{b-a}(e^{-at}-e^{-bt})$
$\dfrac{s}{(s+a)(s+b)}$	$\dfrac{1}{a-b}(ae^{-at}-be^{-bt})$
$\dfrac{s+z}{(s+a)(s+b)}$	$\dfrac{1}{b-a}[(z-a)e^{-at}-(z-b)e^{-bt}]$
$\dfrac{1}{(s+a)(s+b)(s+c)}$	$\dfrac{e^{-at}}{(b-a)(c-a)}+\dfrac{e^{-bt}}{(c-b)(a-b)}+\dfrac{e^{-ct}}{(a-c)(b-c)}$
$\dfrac{s+z}{(s+a)(s+b)(s+c)}$	$\dfrac{(z-a)e^{-at}}{(b-a)(c-a)}+\dfrac{(z-b)e^{-bt}}{(c-b)(a-b)}+\dfrac{(z-c)e^{-ct}}{(a-c)(b-c)}$
$\dfrac{\omega}{s^2+\omega^2}$	$\sin \omega t$
$\dfrac{s}{s^2+\omega^2}$	$\cos \omega t$
$\dfrac{s+z}{s^2+\omega^2}$	$\sqrt{\dfrac{z^2+\omega^2}{\omega^2}}\sin(\omega t+\phi) \qquad \phi \equiv \tan^{-1}(\omega/z)$
$\dfrac{s\sin\phi+\omega\cos\phi}{s^2+\omega^2}$	$\sin(\omega t+\phi)$
$\dfrac{1}{(s+a)^2+\omega^2}$	$\dfrac{1}{\omega}e^{-at}\sin \omega t$

$F(s)$	$f(t) \qquad t > 0$
$\dfrac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2}$	$\dfrac{1}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t \qquad \omega_d \equiv \omega_n \sqrt{1 - \zeta^2}$
$\dfrac{s + a}{(s + a)^2 + \omega^2}$	$e^{-at} \cos \omega t$
$\dfrac{s + z}{(s + a)^2 + \omega^2}$	$\sqrt{\dfrac{(z - a)^2 + \omega^2}{\omega^2}}\, e^{-at} \sin (\omega t + \phi) \qquad \phi \equiv \tan^{-1} \left(\dfrac{\omega}{z - a}\right)$
$\dfrac{1}{s}$	$u(t)$ or 1 unit step
$\dfrac{1}{s} e^{-Ts}$	$u(t - T)$ delayed step
$\dfrac{1}{s} (1 - e^{-Ts})$	$u(t) - u(t - T)$ rectangular pulse
$\dfrac{1}{s(s + a)}$	$\dfrac{1}{a} (1 - e^{-at})$
$\dfrac{1}{s(s + a)(s + b)}$	$\dfrac{1}{ab} \left(1 - \dfrac{be^{-at}}{b - a} + \dfrac{ae^{-bt}}{b - a}\right)$
$\dfrac{s + z}{s(s + a)(s + b)}$	$\dfrac{1}{ab} \left(z - \dfrac{b(z - a)e^{-at}}{b - a} + \dfrac{a(z - b)e^{-bt}}{b - a}\right)$
$\dfrac{1}{s(s^2 + \omega^2)}$	$\dfrac{1}{\omega^2} (1 - \cos \omega t)$
$\dfrac{s + z}{s(s^2 + \omega^2)}$	$\dfrac{z}{\omega^2} - \sqrt{\dfrac{z^2 + \omega^2}{\omega^4}} \cos (\omega t + \phi) \qquad \phi \equiv \tan^{-1} (\omega/z)$
$\dfrac{1}{s(s^2 + 2\zeta\omega_n s + \omega_n^2)}$	$\dfrac{1}{\omega_n^2} - \dfrac{1}{\omega_n \omega_d} e^{-\zeta\omega_n t} \sin (\omega_d t + \phi)$ $\omega_d \equiv \omega_n \sqrt{1 - \zeta^2} \qquad \phi \equiv \cos^{-1} \zeta$
$\dfrac{1}{s(s + a)^2}$	$\dfrac{1}{a^2} (1 - e^{-at} - ate^{-at})$
$\dfrac{s + z}{s(s + a)^2}$	$\dfrac{1}{a^2} [z - ze^{-at} + a(a - z)te^{-at}]$
$\dfrac{1}{s^2}$	t unit ramp
$\dfrac{1}{s^2(s + a)}$	$\dfrac{1}{a^2} (at - 1 + e^{-at})$
$\dfrac{1}{s^n} \qquad n = 1, 2, 3, \ldots$	$\dfrac{t^{n-1}}{(n - 1)!} \qquad 0! = 1$

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B.2 TABLE OF z-TRANSFORMS

$\mathcal{F}(s)$ is the Laplace transform of $f(t)$ and $F(z)$ is the z-transform of $f(nT)$. Unless otherwise noted, $f(t) = 0$, $t < 0$ and the region of convergence of $F(z)$ is outside a circle $r < |z|$ such that all poles of $F(z)$ are inside r.

Table B.2

Number	$\mathcal{F}(s)$	$f(nT)$	$F(z)$
1	—	$1,\ n=0;\ 0\ n\neq 0$	1
2	—	$1,\ n=k;\ 0\ n\neq k$	z^{-k}
3	$\frac{1}{s}$	$1(nT)$	$\frac{z}{z-1}$
4	$\frac{1}{s^2}$	nT	$\frac{Tz}{(z-1)^2}$
5	$\frac{1}{s^3}$	$\frac{1}{2!}(nT)^2$	$\frac{T^2}{2}\frac{z(z+1)}{(z-1)^3}$
6	$\frac{1}{s^4}$	$\frac{1}{3!}(nT)^3$	$\frac{T^3}{6}\frac{z(z^2+4z+1)}{(z-1)^4}$
7	$\frac{1}{s^m}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}e^{-anT}$	$\lim_{a\to 0}\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}$
8	$\frac{1}{s+a}$	e^{-anT}	$\frac{z}{z-e^{-aT}}$
9	$\frac{1}{(s+a)^2}$	nTe^{-anT}	$\frac{Tze^{-aT}}{(z-e^{-aT})^2}$
10	$\frac{1}{(s+a)^3}$	$\frac{1}{2}(nT)^2e^{-anT}$	$\frac{T^2}{2}e^{-aT}\frac{z(z+e^{-aT})}{(z-e^{-aT})^3}$
11	$\frac{1}{(s+a)^m}$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}(e^{-anT})$	$\frac{(-1)^{m-1}}{(m-1)!}\frac{\partial^{m-1}}{\partial a^{m-1}}\frac{z}{z-e^{-aT}}$
12	$\frac{a}{s(s+a)}$	$1-e^{-anT}$	$\frac{z(1-e^{-aT})}{(z-1)(z-e^{-aT})}$

Number	$\mathcal{F}(s)$	$f(nT)$	$F(z)$
13	$\frac{a}{s^2(s+a)}$	$\frac{1}{a}(anT-1+e^{-anT})$	$\frac{z[(aT-1+e^{-aT})z+(1-e^{-aT}-aTe^{-aT})]}{a(z-1)^2(z-e^{-aT})}$
14	$\frac{b-a}{(s+a)(s+b)}$	$(e^{-anT}-e^{-bnT})$	$\frac{(e^{-aT}-e^{-bT})z}{(z-e^{-aT})(z-e^{-bT})}$
15	$\frac{s}{(s+a)^2}$	$(1-anT)e^{-anT}$	$\frac{z[z-e^{-aT}(1+aT)]}{(z-e^{-aT})^2}$
16	$\frac{a^2}{s(s+a)^2}$	$1-e^{-anT}(1+anT)$	$\frac{z[z(1-e^{-aT}-aTe^{-aT})+e^{-2aT}-e^{-aT}+aTe^{-aT}]}{(z-1)(z-e^{-aT})^2}$
17	$\frac{(b-a)s}{(s+a)(s+b)}$	$be^{-bnT}-ae^{-anT}$	$\frac{z[z(b-a)-(be^{-aT}-ae^{-bT})]}{(z-e^{-aT})(z-e^{-bT})}$
18	$\frac{a}{s^2+a^2}$	$\sin anT$	$\frac{z\sin aT}{z^2-(2\cos aT)z+1}$
19	$\frac{s}{s^2+a^2}$	$\cos anT$	$\frac{z(z-\cos aT)}{z^2-(2\cos aT)z+1}$
20	$\frac{s+a}{(s+a)^2+b^2}$	$e^{-anT}\cos bnT$	$\frac{z(z-e^{-aT}\cos bT)}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
21	$\frac{b}{(s+a)^2+b^2}$	$e^{-anT}\sin bnT$	$\frac{ze^{-aT}\sin bT}{z^2-2e^{-aT}(\cos bT)z+e^{-2aT}}$
22	$\frac{a^2+b^2}{s((s+a)^2+b^2)}$	$1-e^{-anT}\left(\cos bnT+\frac{a}{b}\sin bnT\right)$	$\frac{z(Az+B)}{(z-1)(z^2-2e^{-aT}(\cos bT)z+e^{-2aT})}$ $A=1-e^{-aT}\cos bT-\frac{a}{b}e^{-aT}\sin bT$ $B=e^{-2aT}+\frac{a}{b}e^{-aT}\sin bT-e^{-aT}\cos bT$